Math 481  Hwk 3  Solutions

1. a) \( \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \)

\[ \mathbf{A} \mathbf{A}^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} \]

So \( \mathbf{F}: \mathbb{R}^4 \rightarrow \mathbb{R}^3 \)

\((a, b, c, d) \mapsto (a^2 + b^2, ac + bd, c^2 + d^2)\)

b) \( \mathbf{F}_* = \begin{bmatrix} 2a & 2b & 0 & 0 \\ c & d & a & b \\ 0 & 0 & 2c & 2d \end{bmatrix} \)

c) \( \mathbf{F}(\mathbf{A}) = \mathbb{I} \iff \begin{cases} a^2 + b^2 = 1 \\ ac + bd = 0 \\ c^2 + d^2 = 1 \end{cases} \)

Denote the rows of \( \mathbf{F}_* \) by \( \mathbf{r}_1, \mathbf{r}_2 \) and \( \mathbf{r}_3 \).

i.e. \[ \mathbf{F}_* = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix} \]
Note \(1, 2\) and \(3\) \(\Rightarrow\) \(r_1, r_2\) and \(r_3\) are not rows of zeros.

\[r_1 \cdot r_2 = 2ac + 2bd = 0 \quad \text{by} \quad 2\]

\[r_1 \cdot r_3 = 0\]

\[r_2 \cdot r_3 = 2ac + 2bd = 0 \quad \text{by} \quad 2\]

Hence the rows of \(F_x\) are nonzero and orthogonal. It follows that the rank of \(F_x\) is 3. Therefore \(F_x\) is onto for all \(A \in F^{-1}(I)\) and so \(F^{-1}(I) = 0(2)\) is a submanifold of \(R^q\) of dimension 1.
2a) \( \phi_{v,B}(p) = \phi_{v,B}(q) \)

\[ B(\phi(p)) + v = B(\phi(q)) + v \]

\[ B(\phi(p)) = B(\phi(q)) \]

\[ \phi(p) = \phi(q) \]  \( \text{(Since } B \text{ is invertible)} \)

\[ p = q \]  \( \text{(Since } \phi \text{ is one-to-one)} \)

2b) \( \phi_{v,B}(p) = B(\phi(p)) + v \)

\[ \phi(p) = B^{-1}(\phi_{v,B}(p) - v) \]

\[ p = \phi^{-1}(B^{-1}(\phi_{v,B}(p) - v)) \]

\[ \phi_{v,B}(x) = \phi^{-1}(B^{-1}(x - v)) \]

So \( \phi_{v,B}(x) = \phi^{-1}(B^{-1}(x - v)) \)

Now \( \psi \circ \phi^{-1}(B^{-1}(x - v)) \)

Since \((U,d) \) and \((V,\psi) \) are compatible \thetext{.}

Overlap map \( \psi \circ \phi^{-1} \) is smooth
The map \( x \mapsto B^{-1}(x-v) \) is also smooth.

By the chain rule \( 4 \circ \Phi^{-1}_{v,B} \) is smooth.

Now we consider the other composition:

\[
\Phi^{-1}_{v,B} \circ 4^{-1}(x) = B(\Phi^{-1}(4(x))) + v
\]

This is again smooth because \( \Phi \circ 4^{-1} \) is smooth and \( x \mapsto Bx + v \) is smooth.

\[
\Phi_{v,B}(p) = B(\Phi(p)) + v
\]

Set \( v = -B(a(p)) \) to get

\[
\Phi_{v,B}(p) = 0.
\]