1) a) Assume there is a line $L$ which is not contained in $U_1 U U_2 V U_3$.

Let $(x_1, x_2, x_3)$ be a point on $L$ which is different from $(0, 0, 0)$.

Since $L$ is not in $U_1$, we have $x_1 = 0$.

Similarly, $x_2 = 0$ and $x_3 = 0$.

This contradicts our choice of $(x_1, x_2, x_3)$, so no such $L$ exists.

b) $\phi_1(U_1) = \{ (x_2, x_3) \in \mathbb{R}^2 \mid x_1 \neq 0 \}$

$\cup \{ (x_2, x_3) \in \mathbb{R}^2 \}$

$= \mathbb{R}^2$

This is clearly open.

$\phi_1(U_1 U U_2)$

$= \{ (x_2, x_3) \in \mathbb{R}^2 \mid x_1 \neq 0, x_2 \neq 0 \}$

$= \mathbb{R}^2 \setminus y$-axis
For a point \( p = (a, b) \) in \( \mathbb{R}^2 \setminus \text{y-axis} \) the ball of radius \( \frac{a}{2} \) around \( p \) is contained in \( \mathbb{R}^2 \setminus \text{y-axis} \).

i.e.

\[
\begin{array}{c}
\text{So } \phi_1(U \cup U_2) \text{ is open.}
\end{array}
\]

\[\phi_2 \circ \phi_1^{-1} : \mathbb{R}^2 \setminus \text{y-axis} \to \mathbb{R}^2\]

\[
\phi_2 \circ \phi_1^{-1} (u, v) = \phi_2(1, u, v) = \left( \frac{1}{u}, \frac{v}{u} \right)
\]

This is clearly smooth away from \( u = 0 \).
Arguing as in parts (a), (b) and (c) it follows that \( \{(U_i, \phi_i) \}_{i=1,2,3} \) is an atlas for \( \mathbb{R}P^2 \).

If we can show that \( \phi_i : U_i \rightarrow \mathbb{R}^2 \) is one-to-one for \( i=1,2,3 \).

It suffices to show that these maps are invertible.

\[
\phi_1^{-1}(u,v) = \text{the line through the origin and the point } (1, u, v)
\]

\[
\phi_2^{-1}(u,v) = \ldots (u,1,v)
\]

\[
\phi_3^{-1}(u,v) = \ldots (u,v,1)
\]
2) (i) \( U_1 \cup U_2 = S^2 \) since \((1,0,0) \in U_2\)

(ii) to prove that \( \phi_1 \) is one-to-one we compute \( \phi_1^{-1} \)

\[
\phi_1(x_1, x_2, x_3) = \left( \frac{x_2}{1-x_1}, \frac{x_3}{1-x_1} \right) = (u, v).
\]

\[
u^2 + v^2 = \frac{x_2^2 + x_3^2}{(1-x_1)^2} = \frac{1-x_1^2}{(1-x_1)^2} = \frac{1+x_1}{1-x_1}
\]

\[\Rightarrow x_1 = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}.
\]

Now \( 1-x_1 = 1 - \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} = \frac{2}{u^2 + v^2 + 1} \)

So \[
x_2 = \frac{u}{1-x_1}, \Rightarrow x_2 = \frac{2u}{u^2 + v^2 + 1},
\]

and \[
x_3 = \frac{v}{1-x_1}, \Rightarrow x_3 = \frac{2v}{u^2 + v^2 + 1},
\]

So \( \phi_1^{-1}(u,v) = \left( \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}, \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1} \right) \).
A similar computation yields a formula for \( \phi_2^{-1} \) and confirms that \( \phi_2 \) is one-to-one.

(iii) \[
\phi_1(u_1) = \left\{ \left( \frac{x_2}{1-x_1}, \frac{x_3}{1-x_1} \right) \mid x_1 \neq 1 \right\} = \mathbb{R}^2.
\]

\[
\phi_1(u_1 \cap U_2) = \left\{ \left( \frac{x_2}{1-x_1}, \frac{x_3}{1-x_1} \right) \mid x_1 \neq \pm 1 \right\} = \mathbb{R}^2 \setminus (0,0).
\]

Both these sets are open.

(iv) \[
\phi_2 \circ \phi_1^{-1}(u,v) = \phi_2 \left( \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}, \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1} \right)
\]

\[
= \left( \frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2} \right).
\]

This is smooth away from \((u,v) = (0,0)\).

Hence \( \{ (U_i, \phi_i) \}_{i=1,2} \) is a smooth atlas for \( S^2 \).