1. Let \( f: S^1 \to \mathbb{R} \) be a smooth function on the circle. In this question you will show that the 1-form \( df \) vanishes at at least one point in \( S^1 \). In other words, there is a point \( q \) in \( S^1 \) such that \( (df)_q: T_qS^1 \to \mathbb{R} \) takes every tangent vector \( V \in T_qS^1 \) to zero. To prove this we will use the following

**FACT:** Every function \( f: S^1 \to \mathbb{R} \) attains its maximum value at some point in \( S^1 \). In other there is a point \( q \in S^1 \) such that \( f(q) \geq f(p) \) for all points \( p \in S^1 \).

Let \( q \) be a point at which \( f \) takes its maximum value and let \( V \) be any element of \( T_qS^1 \). Prove that \( (df)_q(V) = 0 \).

Hint: Let \( \gamma: \mathbb{R} \to S^1 \) be a curve such that \( \gamma(0) = q \) and \( \gamma_*(1) = V \). Recall that

\[
(df)_q(V) = \frac{d}{dt}_{t=0} (f(\gamma(t))).
\]

Prove that this is zero by considering the behavior of the function of one variable \( f(\gamma(t)) \) near \( t = 0 \).

2. In class, we defined a 1-form \( \alpha \) on \( S^1 \) by defining its coordinate representatives for the usual charts \( \{(U_i, \phi_i)\}_{i=1,...,4} \) as follows:

\[
\begin{align*}
\alpha^{U_1}(x) &= \frac{dx}{\sqrt{1-x^2}} & \alpha^{U_2}(x) &= \frac{dx}{\sqrt{1-x^2}} \\
\alpha^{U_3}(y) &= \frac{dy}{\sqrt{1-y^2}} & \alpha^{U_4}(y) &= -\frac{dy}{\sqrt{1-y^2}}.
\end{align*}
\]

(a) Show that \( \alpha \) is indeed a 1-form by checking that

\[
\alpha^{U_i} = \alpha^{U_j} \circ (\phi_j \circ \phi_i)
\]

for all relevant \( i \neq j \). (We already did this for \( i = 3 \) and \( j = 1 \) in class.)

(b) Show that \( \alpha \) never vanishes. In other words, show that none of the maps \( \alpha(p): T_pS^1 \to \mathbb{R} \) take every tangent vector to zero.
(c) Conclude from (1) and (2b) that $\alpha$ is not equal to $df$ for any function $f : S^1 \to \mathbb{R}$.

3. Let $E$ be a 2-dimensional vector space with basis $\{e_1, e_2\}$. Let $\{\sigma^1, \sigma^2\}$ be the dual basis of $E^*$. Show that the $\left(\begin{array}{c}1 \\ 1 \end{array}\right)$-tensors

$$e_1 \otimes \sigma^1, \ e_1 \otimes \sigma^2, \ e_2 \otimes \sigma^1 \text{ and } e_2 \otimes \sigma^2$$

are linearly independent.

4. Consider the charts $(U, \phi)$ and $(V, \psi)$ on $\mathbb{R}P^2$ where

$$U = \{[x : y : z] \in \mathbb{R}P^2 \mid z \neq 0\}, \quad \phi([x : y : z]) = \left(\frac{x}{z}, \frac{y}{z}\right) = (u_1, u_2)$$

and

$$V = \{[x : y : z] \in \mathbb{R}P^2 \mid y \neq 0\}, \quad \psi([x : y : z]) = \left(\frac{x}{y}, \frac{z}{y}\right) = (w_1, w_2).$$

Let $A$ be a $\left(\begin{array}{c}1 \\ 1 \end{array}\right)$-tensor field on $\mathbb{R}P^2$ such that in the chart $(U, \phi)$ we have

$$A^U(u_1, u_2) = ((u_1)^2u_2) \frac{\partial}{\partial u_1} \otimes du_2.$$ 

Compute $A^V$ on $U \cap V$. 

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