1. \((O(2) \text{ revisited })\)

In class we realized \(O(2)\) as \(F^{-1}(I)\) for the map \(F: M(2) \rightarrow Sym(2)\) defined by \(F(A) = AA^T\). We then proved that \(O(2)\) is a manifold by showing that the identity matrix \(I\) is a regular value of \(F\). In this exercise we will prove that \(I\) is a regular value in a different way.

(a) Identify \(M(2)\) with \(\mathbb{R}^4\) and \(Sym(2)\) with \(\mathbb{R}^3\), and write down the mapping \(F\) in these global coordinates.

(b) Compute the differential \(F^*\) of \(F\) at \(A\) as a matrix.

(c) Show that if \(F(A) = I\) then \(F^*\) has rank 3. (Hint: the rows of \(F^*\) should be orthogonal to one another.) Conclude that \(I\) is a regular value of \(F\).

2. Let \(M\) be an \(m\)-dimensional manifold and consider two compatible charts \((U, \phi)\) and \((V, \psi)\) on \(M\). Consider a vector \(v \in \mathbb{R}^m\), and an invertible \(m \times m\) matrix \(B\).

Define the map \(\phi_{v,B}: U \rightarrow \mathbb{R}^m\) by

\[
\phi_{v,B}(p) = B(\phi(p)) + v.
\]

(a) Show that \(\phi_{v,B}\) is one-to-one onto it’s image.

(b) Show that \((U, \phi_{v,B})\) is compatible with \((V, \psi)\).

(c) Find a vector \(v\) so that \(\phi_{v,B}(p) = 0\).

3. (Extra Problem for those students taking the course for 4 credits or for honors credit)

Let \(F: M \rightarrow M\) be a smooth mapping whose differential at \(p \in M\) is one-to-one. Using the ideas of Question 2, show that there are charts \((U, \phi)\) and \((V, \psi)\) around \(p\) and \(F(p)\), respectively, such that

\[
(\psi \circ F \circ \phi^{-1})_* = I.
\]

4. (Extra Problem for those students taking the course for 4 credits or for honors credit)

Let \(SL(n)\) be set of \(n \times n\) matrices whose determinant is equal to 1. Prove that \(SL(n)\) is a manifold.