Lecture 5  Let’s start w/ an observation.

Consider:

\[ \mathbb{R}^2 = \{ (x, y) \mid x, y \in \mathbb{R} \} \]

(\text{the vector space})

Consider a line through the origin:

\[ L = \{ (x, y) \in \mathbb{R}^2 \mid y = mx \} = \{ (x, xm) \mid x \in \mathbb{R} \} \]

If you add two vectors on the line, the result is on the line:

\[ (x_1, x_1m) + (x_2, x_2m) = (x_1 + x_2, x_1m + x_2m) = (x_1 + x_2, (x_1 + x_2)m) \]

If you multiply a vector on \( L \) by a scalar the result is still on \( L \).

So, the operations preserve \( L \) and it looks like \( L \) inherits the structure of a vector space from \( \mathbb{R}^2 \).

**Def**  Suppose \( V \) is a vector space.

A subset \( W \subseteq V \) is a **subspace** if
(a) $\overline{0}$ is in $W$

(b) For all $w_1, w_2 \in W$, $w_1 + w_2 \in W$
(W is closed under addition)

(c) For all $c \in \mathbb{R}$ and $w \in W$, $cw \in W$
(W is closed under scalar multiplication)

Thus If $W$ is a subspace of $V$, then $W$ is itself a vector space for the inherited addition and scalar multiplication.

Proof

Given (17-8) for $V$ and (a), (b), (c) for $W$

need (17-8) for $W$ (call these (1)_w - (8)_w).

Now (1)_w - (2)_w and (5)_w - (8)_w follow from corresponding properties of $V$. (the are restrictions of these facts to $W$)

(3)_w follows from (a) \[ \overline{0}_W = \overline{0}_V \]

That leaves (4)_w. 

Given $v \in W$ it follows from (9) that $v \in V$
such that $v + w = \overline{0}$. For (4)_w need $v \in W$.

But recall $v = (1)_w v$ so $v \in W$ by (c).
**Remark** to show \( W \) is a vector space it suffices to recognize it as a subspace of a vector space \( V \) (check (a), (b), (c))

**Exercise**

Show \( W = \{ (t_1 + t_2, t_1 - t_2, t_1) | t_1, t_2 \in \mathbb{R}^3 \} \) with componentwise addition and scalar multipl has is a vector space.

**Proof** \( W \subset \mathbb{R}^3 \) and it inherits addition and scalar multiplication from \( \mathbb{R}^3 \)

Just need to check (a), (b), and (c).

(a): \( t_1 = 0, t_2 = 0 \Rightarrow (0, 0, 0) = 0 \in \mathbb{R}^3 \)

(b): \( (t_1 + t_2, t_1 - t_2, t_1) + (s_1 + s_2, s_1 - s_2, s_1) \)

\[= (t_1 + t_2 + s_1 + s_2, t_1 - t_2 + s_1 - s_2, t_1 + s_1) = (t_1 + s_1) + (t_2 + s_2), (t_1 + s_1) - (t_2 + s_2), (t_1, 15, 2) \in W \]

(c) **Exercise 1**

**Remark** this looks like standard form of a solution set of a linear system

**Remark 2** \( W' = \{ (t_1 + t_2, t_1 - t_2, t_1 + 1) | t_1, t_2 \in \mathbb{R}^3 \} \) is not a subspace of \( \mathbb{R}^3 \) since \( 0 \not\in W' \) (exercise 2).
Example: \( P_k \) is a subspace of \( P_{k+1} \).

Example: \( C^0(\mathbb{R}) \) is a subspace of \( \mathcal{F}(\mathbb{R}) \).

(a): \( 0 \cdot x = 0 \) is continuous (constant function)

(b): \( f, g \) cont \( \Rightarrow f+g \) cont

(c): \( f \) cont \( \Rightarrow cf \) cont.

Example: Consider the following map

\[ \text{transpose}: \ M_{n \times n} \rightarrow M_{n \times n} \]

\[ A = (a_{ij}) \mapsto A^t = (a_{ji}) \]

Example:

\[
\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}
\]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}
\]

Definition: \( A \in M_{n \times n} \) is symmetric if \( A = A^t \).

Claim: The subset of symmetric matrices in \( M_{n \times n} \) is a subspace.

Proof (a): \( 0 \) is symmetric \( (a_{ij} = a_{ji} = 0) \).
(b): \( A = (a_{ij}) \), \( B = (b_{ij}) \) symmetric

\[ a_{ij} = a_{ji} \quad \text{and} \quad b_{ij} = b_{ji} \]

\[ a_{ij} + b_{ij} = a_{ji} + b_{ji} \]

\[ A + B = (A + B)^+ \]

(c): exercise 3.

\textbf{Thm} If \( W_1 \) and \( W_2 \) are subspace of \( V \) then so is \( W_1 \cap W_2 \).

\textbf{Pf} (a): \( \overline{0} \in W_1 \) and \( \overline{0} \in W_2 \) so \( \overline{0} \in W_1 \cap W_2 \).

(b): given \( u,v \in W_1 \cap W_2 \)

\[ u + v \in W_1 \quad \text{and} \quad u + v \in W_2 \]

so \( u + v \in W_1 \cap W_2 \)

(c): exercise 4