Math 416  Lecture 1

Welcome to Abstract Linear Algebra.

Today: an overview of the objects we will study.

The most concrete objects will be matrices.

ex. \((1 \ 0)\), \((1 \ 2 \ 3)\), \(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\), ...

A general \(m \times n\) matrix will be denoted as

\[
A = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} = (a_{ij})
\]

Here, the entries \(a_{ij}\) are real numbers.

\(m = \# \text{ rows}\)

\(n = \# \text{ columns}\)

We will do algebra with matrices and put them in various simplified forms.

We will use them in different ways and will compare these different perspectives.
The most general (abstract) objects we will study are linear transformations (maps) of vector spaces, \( L : V \rightarrow W \).

Examples of vector spaces.

1) \( \mathbb{R}^2 = \{ (x, y) \mid x, y \in \mathbb{R} \} \)
   - can add \((x, y) + (x', y') = (x + x', y + y')\)
   - and multiply \(c(x, y) = (cx, cy)\).

These operations have geometric interpretations.

2) \( \mathbb{R}^n = \{ (a_1, a_2, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathbb{R} \} \)

2') \( \mathbb{R}^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \mid a_1, \ldots, a_n \in \mathbb{R} \right\} \)

3) \( C^0(\mathbb{R}) = \{ \text{continuous functions} \} \)
\[ f(x) = x^2 \quad g(x) = \sin 2\pi x. \]

\[ (f + g)(x) = x^2 + \sin 2\pi x \]

\[ (3f)(x) = 3x^2 \]

Examples of linear maps:

1) \[ L : \mathbb{R}^2 \rightarrow \mathbb{R} \]
   \[ (x, y) \mapsto 3x + \pi y \]

2) \[ L : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]
   \[ (x, y) \mapsto \left( \frac{3}{7}x + 2y, \frac{3}{7}x + 4y \right) \]

3) Reflection \[ : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]
   \[ (x, y) \mapsto (y, x) \]
4) \( \text{Diff} : V \rightarrow W \)

\[ W = C^\infty(\mathbb{R}) \]

\[ V = \{ \text{continuously differentiable functions} \} = C^1(\mathbb{R}) \]

\( \text{Diff}(f) = f' \) is also a linear map!

5) An \( m \times n \) matrix \( A \) determines a linear map from \( \mathbb{R}^n \) to \( \mathbb{R}^m \)

\[ \text{ex} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

\[ \mapsto L_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]
\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}
\]

Studying these linear maps is deeply related to solving systems of linear equations. We will start our real studies here.

Systems of Linear equations

Variables: \( x_1, \ldots, x_n \) \( (n \text{ of them}) \)

Equations \( (m \text{ of them}) \)

\[
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \quad \text{\#1} \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \quad \text{\#2} \\
\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m \quad \text{\#m}
\]

\( a_{ij}, b_i \) are fixed real numbers.

To solve the system we want to find all \( n \)-tuples \( (x_1, \ldots, x_n) \) in \( \mathbb{R}^n \) which satisfy all \( m \)-equations simultaneously.