INSTRUCTIONS: You must show all of your work to receive credit. Please write your answers in the space provided. No calculators, electronics, notes or other aids are to be used while taking this test.
(1) (9 points)

(a) Give a formula for the determinant of a matrix \( A = (a_{ij}) \in M_{n \times n}(F) \). Make sure to define all the relevant terms in your formula.

\[
\text{det}(A) = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \text{det}(\hat{A}_{ij})
\]

\( \hat{A}_{ij} \) is the element of \( M_{(n-1) \times (n-1)}(F) \) obtained from \( A \) by deleting its \( i \)-th row and \( j \)-th column.

(b) Let \( V \) be a finite dimensional vector space. Let \( \beta \) and \( \beta' \) be two ordered bases for \( V \) and consider a map \( T \in \mathcal{L}(V, V) \). Give the change-of-basis formula which relates \( [T]_{\beta}^{\beta} \) and \( [T]_{\beta'}^{\beta} \). Take care again to define all the relevant terms in your formula.

\[
[T]_{\beta'}^{\beta} = Q^{-1} [T]_{\beta}^{\beta} Q \quad \text{where}
\]

\[
Q = [I]_{\beta}^{\beta'}
\]

(c) Give the statement of the Cayley-Hamilton Theorem.

Suppose \( \dim(V) = n \) and \( T \in \mathcal{L}(V, V) \).

If \( f(t) \) is the characteristic polynomial of \( T \), then

\[
f(T) = 0 \in \mathcal{L}(V, V)
\]
(2) (13 points)

(a) (10 points) Compute the determinant of the matrix \( A = \begin{pmatrix} -4 & 5 & -10 & -6 \\ 2 & -1 & -1 & 4 \\ \pi & -\pi & 2\pi & \pi \\ 3 & -2 & 10 & -1 \end{pmatrix} \).

\[
\det(A) = -\pi \det \begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{pmatrix}
= -\pi \det \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -9 \\ 0 & 0 & 9 & -6 \end{pmatrix}
= -\pi (1)(1)(3)(6) = -\pi 18
\]

(b) (3 points) Compute the determinant of the matrix \( B = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & -4 & 5 & -10 & -6 \\ 0 & 2 & -1 & -1 & 4 \\ 0 & \pi & -\pi & 2\pi & \pi \\ 0 & 3 & -2 & 10 & -1 \end{pmatrix} \).

\[
\det(B) = \sum_{i=1}^{5} b_{ij} (-1)^{i+j} \det(\tilde{B}_{ij})
= b_{11} (-1)^{2} \det(\tilde{B}_{11})
= \sqrt{2} \det(\tilde{C})
= -\pi 18 \sqrt{2}
\]
(3) (18 points) Consider the linear map $T: P^2(\mathbb{R}) \to P^2(\mathbb{R})$ defined by 

$$T(a + bx + cx^2) = (a + b + c) + (b + 2c)x + (3c)x^2.$$  

(a) Determine if $T$ is invertible. Let $\beta = \{1, x, x^2\}$

$$
T(1) = 1 \\
T(x) = 1 + x \\
T(x^2) = 1 + 2x + 3x^2
$$

$\Rightarrow [T]_\beta^\beta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

Let $( [T]_\beta^\beta)^* = 3 \neq 0 \Rightarrow [T]_\beta^\beta$ is invertible $\Rightarrow T$ is invertible

(b) Find the eigenvalues of $T$ and determine their algebraic multiplicities.

$$
\det ([T]_\beta^\beta - t I_3) = \det \begin{pmatrix} 1-t & 1 & 1 \\ 0 & 1-t & 2 \\ 0 & 0 & 3-t \end{pmatrix} = (1-t)^2 (3-t)
$$

So $\lambda_1 = 1$ is an eigenvalue with alg mult. 2.

and $\lambda_2 = 3$ is an eigenvalue with alg mult 1

(c) Determine if $T$ is diagonalizable.

$T$ is diagonalizable if geom mult of $\lambda_1$ is 2

$E_{\lambda_1} = N(T - I_3)$ and $\dim(E_{\lambda_1}) = \dim(N([T]_\beta^\beta - I_3))$

$$
[T]_\beta^\beta - I_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}
$$

has 1 - nonpivot column so

$\dim(E_{\lambda_1}) = 1 \neq 2$ and

$T$ is not diagonalizable.
(4) (18 points) Compute $A^{136}$ where $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Your final answer should be in the form of a single $2 \times 2$ matrix. In fact, when you are done please rewrite your answer here:

$$A^{136} = \begin{pmatrix} \frac{1}{2} \left( 3^{136} + 1 \right) & \frac{1}{2} \left( 3^{136} - 1 \right) \\ \frac{1}{2} \left( 3^{136} - 1 \right) & \frac{1}{2} \left( 3^{136} + 1 \right) \end{pmatrix}.$$

Let $(A - tI_2) = \begin{pmatrix} 2-t & 1 \\ 1 & 2-t \end{pmatrix}$.

$(2-t)^2 - 1 = t^2 - 4t + 3 = (t - 3)(t - 1)$

So $A$ has distinct eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$ and is diagonalizable.

$\lambda_1 = 3$: $A - 3I_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ $\sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ so $(1, 1)$ is an eigenvector.

$\lambda_2 = 1$: $A - I_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $(1, -1)$ is an eigenvector.

Setting $\Theta' = \left\{ (1, 1), (1, -1) \right\}$ we have

$Q = [I_v]_{\Theta'}^\Theta = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $Q^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and

$Q^{-1} A Q = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

So $A^{136} = Q \begin{pmatrix} 3^{136} & 0 \\ 0 & 3^{136} \end{pmatrix} Q^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^{136} & 0 \\ 0 & 3^{136} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{2} \begin{pmatrix} 3^{136} & -1 \\ 3^{136} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{2} \begin{pmatrix} 3^{136} + 1 & 3^{136} - 1 \\ 3^{136} - 1 & 3^{136} + 1 \end{pmatrix}$.
(5) (10 points) Do ONE of the following THREE questions. Circle which of the questions
(a)  (b)  (c)
you would like graded.

(a)  (i) Find the $3 \times 3$ elementary matrix $E$ which corresponds to the elementary row
operation of interchanging the first and third rows of a $3 \times 3$ matrix.

$$E = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix} \text{ (apply row operation to } \mathbf{I}_3 \text{)}$$

(ii) Compute the determinant of $E$.

$$\det(E) = (-1)^{H_3(1)} \det \left( \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \right) = -1$$

(iii) Use $E$ and the properties of the determinant to prove that if $B$ is obtained from
$A \in M_{3\times3}(F)$ by interchanging its first and third rows, then $\det(B) = -\det(A)$.

$$B = EA$$

$$\Rightarrow \quad \det(B) = \det(EA) = \det(E) \det(A) = -\det(A)$$

(b) Let $V$ be a vector space of dimension $n$ and consider a map $T \in \mathcal{L}(V, V)$.

(i) Define the characteristic polynomial of $T$.

$$\det \left( \left[ T \right]_\mathbf{B}^\mathbf{B} - t \mathbf{I}_n \right) \text{ where } \mathbf{B} \text{ is a basis of } V$$
(ii) Show that your definition of the characteristic polynomial does not depend on the choice of basis you have made.

\[
\det \left( [T]_{v'} - t I_n \right) = \det \left( Q^{-1} \left[ T \right]_v Q - t I_n \right) \\
= \det \left( Q^{-1} \left[ T \right]_v - t I_n \right) Q \\
= \det \left( Q^{-1} \right) \det \left( [T]_v - t I_n \right) \det \left( Q \right) \\
= \det \left( [T]_v - t I_n \right).
\]

(c) Let \( V \) be a finite dimensional vector space and consider a map \( T \in \mathcal{L}(V, V) \).

(i) Define what it means for a subspace \( W \) of \( V \) to be \( T \)-invariant.

\[
T(W) \subset W
\]

(ii) For \( x \in V \) give the definition of \( W_x \), the \( T \)-cyclic subspace of \( V \) generated by \( x \).

\[
W_x = \text{Span} \left\{ x, T(x), T^2(x), \ldots \right\}
\]

(iii) Show that \( W_x \) is a \( T \)-invariant subspace of \( V \).

\[
V \ni W_x \\
\Rightarrow \quad V = a_0 x + a_1 T(x) + \ldots + a_n T^n(x) \\
\Rightarrow \quad T(V) = a_0 T(x) + a_1 T^2(x) + \ldots + a_n T^{n+1}(x) \\
\Rightarrow \quad T(v) \in W_x
\]
Bonus Problems (6 points)

B1: (4 points)
Prove that $\lambda$ is an eigenvalue of $A \in M_{n \times n}(F)$ if and only if $\det(A - \lambda I_n) = 0$.

$\lambda$ is eigenvalue of $A$ $\iff$ $Av = \lambda v$ for $v \neq 0 \in M_{n \times 1}(F)$
$\iff (A - \lambda I_n)(v) = 0$ for $v \neq 0 \in M_{n \times 1}(F)$
$\iff \text{rank } (A - \lambda I_n) < n$
$\iff A - \lambda I_n \text{ is not invertible}$
$\iff \det(A - \lambda I_n) = 0$

B2: (2 points)
Show that the assertion of the Caley-Hamilton Theorem holds for the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

from Question 4.

From Q4 we have char poly $f(t) = t^2 - 4t + 3$.

Caley-Hamilton $\Rightarrow$ $A^2 - 4A + 3I_2 = 0 \in M_{2 \times 2}$

Check:

$$A^2 - 4A + 3I_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 4 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$