
To use the integral test, you should check that the function \( f(x) \) in question is positive, decreasing, and has \( \lim_{x \to \infty} f(x) = 0 \). Remember that you typically only need to check whether or not \( \int_1^\infty f(x) \, dx \) converges or diverges (i.e. you do not have to evaluate the integral).

1. Use the integral test to determine if \( \sum_{n=1}^{\infty} \frac{n^2}{e^n} \) converges or diverges.

\[ f(x) = \frac{x^2}{e^x} \text{ positive, decreasing, continuous for } x \geq 2, \quad \lim_{x \to \infty} f(x) = 0 \]

\[ \int_1^\infty \frac{x^2}{e^x} \, dx \text{ conv. by comp. test } [0 < \frac{x^2}{e^x} \leq \frac{x^2}{e^{100}} \text{ for large } x \text{ & } \int_1^\infty \frac{x^2}{e^{100}} \, dx \text{ conv.}] \]

By Integral Test, \( \sum_{n=1}^{\infty} \frac{n^2}{e^n} \) conv.

2. Use the integral test to determine if \( \sum_{n=1}^{\infty} \frac{n}{n^3 + 2} \) converges or diverges.

**HINT:** It is possible, but tedious, to do the integral directly. Try to test convergence of the integral with a comparison test. \( f'(x) = \frac{2x(x^3+2)}{(x^3+2)^2} \leq 0 \text{ for } x \geq 1 \)

\[ f(x) = \frac{x}{x^3+2} \text{ positive, decreasing, continuous for } x \geq 1, \quad \lim_{x \to \infty} f(x) = 0 \]

\[ \int_1^\infty \frac{x}{x^3+2} \, dx \text{ conv. by comp. test } [0 \leq \frac{x}{x^3} \leq \frac{x}{x^2} = \frac{1}{x} \text{ & } \int_1^\infty \frac{1}{x^2} \, dx \text{ conv.}] \]

So, \( \sum_{n=1}^{\infty} \frac{n}{n^3+2} \) conv. by Integral Test.

3. Use the integral test to determine if \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)} \) converges or diverges.

\[ f(x) = \frac{1}{x \ln x} \text{ positive, decreasing, continuous for } x \geq 2, \quad \lim_{x \to \infty} f(x) = 0 \]

\[ \int_2^\infty \frac{1}{x \ln x} \, dx = \left( \int_2^\infty \frac{1}{u} \, du \right) \text{ div. (p=1) } \implies \sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ div. by Integral Test.} \]
4. Use the integral test to show that \( \sum_{n=2}^{\infty} \frac{1}{n(ln n)^p} \) converges for \( p > 1 \). What does problem (3) tell you about what happens when \( p \leq 1 \)?

By Integral Test,

\[
\int_{2}^{\infty} \frac{1}{x(ln x)^p} \, dx = \lim_{u \to \infty} \int_{ln 2}^{u} \frac{1}{u^p} \, du
\]

conv. for \( p > 1 \) \( \Rightarrow \)

\[
\sum_{n=2}^{\infty} \frac{1}{n(ln n)^p}
\]

conv. for \( p > 1 \)

When \( p \leq 1 \),

\[
0 \leq \frac{1}{x(ln x)^p} \leq \frac{1}{x(ln x)}
\]

\( \Rightarrow \)

\[
\sum_{n=1}^{\infty} \frac{1}{n(ln n)^p}
\]

div. by Integral Test.

5. Recall that the harmonic series is \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \). Let \( s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \) be the \( n \)th partial sum.

a) What is \( \lim_{n \to \infty} s_n \)? (Recall that the harmonic series diverges.)

\[
\lim_{n \to \infty} s_n = \infty
\]

b) Draw a careful picture on the graph to the right which illustrates that \( s_n \geq ln(n+1) \). Be sure that your reasoning is explained.

c) Draw a (different) careful picture on the graph to the right which illustrates that \( s_n \leq 1 + ln \, n \). Be sure that your reasoning is explained.

\[
S_n = \text{sum of } s_n \leq 1 + \int_{1}^{n} \frac{1}{x} \, dx = 1 + ln \, n
\]

\[
\text{area under } y = \frac{1}{x} \text{ from } 1 \text{ to } (n+1)
\]

\[
\text{area under } y = \frac{1}{x} \text{ from } 1 \text{ to } n
\]

\[
S_{3,000,000,000} \leq 1 + ln \left( 3,000,000,000 \right) \approx 22.82 \leq 23
\]