
1. This problem outlines the derivation of the "Test for divergence", which states that if \( \lim_{n \to \infty} a_n \neq 0 \) then the series \( \sum a_n \) must diverge. Recall that for a series \( \sum a_n \) the partial sums \( S_n \) are defined as follows:

\[
S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n
\]

a) Express \( a_n \) in terms of \( S_n \) and \( S_{n-1} \).

\[
\begin{align*}
S_n &= a_1 + a_2 + \ldots + a_{n-1} + a_n \\
\Rightarrow \quad S_n - S_{n-1} &= a_n
\end{align*}
\]

b) Suppose that \( \sum_{n=1}^{\infty} a_n = S \). Find \( \lim_{n \to \infty} S_n \) and \( \lim_{n \to \infty} S_{n-1} \).

\[
\lim_{n \to \infty} S_n = S = \lim_{n \to \infty} S_{n-1}
\]

c) Suppose that \( \sum_{n=1}^{\infty} a_n \) converges to a value \( S \). Find \( \lim_{n \to \infty} a_n \).

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1}) = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} = S - S = 0
\]

2. Determine convergence of the following series

(a) \( \sum_{n=0}^{\infty} \frac{n^2}{n^2 + 1} \) div. by Test for divergence \( \left( \lim_{n \to \infty} \frac{n^2}{n^2 + 1} = 1 \neq 0 \right) \)

(b) \( \sum_{n=0}^{\infty} \frac{e^n}{3^n} \) conv. \( \left( \text{geometric series } r = \frac{e}{3} \quad |r| < 1 \right) \)

(c) \( \sum_{n=0}^{\infty} e^{-\frac{n}{n+1}} \) div. by Test for div. \( \left( \lim_{n \to \infty} e^{-\frac{n}{n+1}} = e^0 = 1 \neq 0 \right) \)

(d) \( \sum_{n=0}^{\infty} \cos \left( \frac{n\pi}{2} \right) \) div. by Test for div. \( \left( \lim_{n \to \infty} \cos \left( \frac{n\pi}{2} \right) \text{ DNE } \neq 0 \right) \)
3. Given $S_n = \frac{n}{n+1}$, find $a_n$ and $\sum_{n=1}^{\infty} a_n$.

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n}{n+1} = 1$$

$$a_1 = S_1 = \frac{1}{2}$$

$n > 1$

$$a_n = S_n - S_{n-1} = \frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2(n+1)(n-1)}{n(n+1)} = \frac{1}{n(n+1)}$$

4. The Sierpinski Carpet is a fractal image which is formed as follows: Start with a square with side length 1. Remove the center $1/9$. Then remove the center $1/9$ from each of the 8 remaining squares. Repeat with each of the smaller squares. Here is a picture of the first three steps, and one from a few steps later. Find the total area of all of the squares which are removed. To do this you should find the pattern which allows you to express the total area as a geometric series, and then find the sum of the series. Why is your answer surprising?

$$A = \frac{1}{9} + \frac{1}{9} \left( \frac{8}{9} \right) + \frac{1}{9} \left( \frac{8}{9} \right)^2 + \frac{1}{9} \left( \frac{8}{9} \right)^3 + \ldots$$

$$= \frac{1}{9} \left( 1 + \frac{8}{9} + \left( \frac{8}{9} \right)^2 + \ldots \right)$$

$$= \frac{1}{9} \cdot \frac{1}{1 - \frac{8}{9}}$$

$$= \frac{9}{1} \cdot \frac{1}{1 - \frac{8}{9}}$$

$$= \frac{9}{1} \cdot \frac{9}{1}$$

$$= 9$$

$|r| < 1$