If you’re given an infinite series \( \sum_{n=1}^{\infty} a_n \) and told to find whether it converges or diverges, but you don’t know how to start, try following these steps.

1. First make sure that you’re actually being asked to determine convergence or divergence of a SERIES. If you’re just being asked to determine convergence or divergence of the SEQUENCE \( \{a_n\} \), then all you need to do is calculate \( \lim_{n \to \infty} a_n \).
   - If \( \lim_{n \to \infty} a_n \) is a finite number \( L \), then say \( \{a_n\} \) converges to \( L \).
   - If \( \lim_{n \to \infty} a_n \) does not exist or is \( \infty \) or \( -\infty \), then say \( \{a_n\} \) diverges.

2. Calculate \( \lim_{n \to \infty} a_n \).
   - If \( \lim_{n \to \infty} a_n \neq 0 \), say \( \sum_{n=1}^{\infty} a_n \) diverges by the Test for Divergence.
   - If \( \lim_{n \to \infty} a_n = 0 \), keep working.

3. Write out the first few terms in the infinite sum, so \( \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots \). Are you multiplying by the same number each time to get from one \( a_n \) to the next?
   - If yes, it’s a geometric series. Write it in the form \( a(1 + r + r^2 + r^3 + \cdots) \).
     - If \( |r| < 1 \), say \( \sum_{n=1}^{\infty} a_n \) converges to \( \frac{a}{1-r} \) by the geometric series test.
     - If \( |r| \geq 1 \), say \( \sum_{n=1}^{\infty} a_n \) diverges by the geometric series test.
   - If no, keep working.

4. Is the series of the form \( \sum_{n=1}^{\infty} \frac{1}{n^p} \)?
   - If yes, find the value of \( p \).
     - If \( p > 1 \), say \( \sum_{n=1}^{\infty} a_n \) converges by the p-test.
     - If \( p \leq 1 \), say \( \sum_{n=1}^{\infty} a_n \) diverges by the p-test.
   - If no, keep working.

5. Does the series look telescoping or can the \( a_n \)’s be rewritten in a way that shows cancellation?
   - If yes, then look at a partial sum \( s_N = \sum_{n=1}^{N} a_n \). Write out the first few terms and the last few terms of \( s_N \), so \( s_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1} + a_N \). Because the series is telescoping, there will be a good deal of cancellation in \( s_N \) and you can get a nice, clean formula for \( s_N \). Then take \( \lim_{N \to \infty} s_N \).
If $\lim_{N \to \infty} s_N$ is a finite real number $S$, say $\sum_{n=1}^{\infty} a_n$ converges to $S$ by telescoping series.

If $\lim_{N \to \infty} s_N$ is not a finite real number, say $\sum_{n=1}^{\infty} a_n$ diverges by telescoping series.

• If no, keep working.

6. Are all the $a_n$’s positive? If so, can you easily compare the $a_n$’s to some $b_n$’s, where you know the behavior of $\sum_{n=1}^{\infty} b_n$?

• If yes, try to use the Comparison Test.
  - If $0 \leq a_n \leq b_n$ for all $n$, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges by Comparison Test.
  - If $0 \leq b_n \leq a_n$ for all $n$, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges by Comparison Test.

• If no, keep working.

7. Are all the $a_n$’s positive? If so, consider what does $a_n$ look like as $n \to \infty$? Specifically, consider which term(s) will dominate in the numerator and which term(s) will dominate in the denominator. Let $b_n$ be defined as the ratio of these dominating terms. Do the $b_n$’s look like the terms of a geometric series, a $p$-series, or any series for which you know convergence or divergence?

• If yes, try to use the Limit Comparison Test. Calculate $\lim_{n \to \infty} \frac{a_n}{b_n}$.
  - If this limit is a positive, finite number then $\sum_{n=1}^{\infty} a_n$ behaves the same way as $\sum_{n=1}^{\infty} b_n$ by Limit Comparison Test.
  - If not, make sure you chose your $b_n$’s correctly.

• If no, keep working.

8. Is the series alternating or can it be rewritten as an alternating series?

• If yes, write the series as $\sum_{n=1}^{\infty} (-1)^n b_n$ and try to use the Alternating Series Test.
  - Verify that the $b_n$’s are decreasing.
  - Verify that $b_n \geq 0$.
  - Verify that $\lim_{n \to \infty} b_n = 0$.
  - If all of these conditions hold, say $\sum_{n=1}^{\infty} a_n$ converges by Alternating Series Test.
  - If one of these conditions does not hold, keep working.

• If no, keep working.

9. Is each factor of $a_n$ raised to a power of $n$? If so, try the Root Test. Calculate $\lim_{n \to \infty} |a_n|^{1/n}$.

• If $\lim_{n \to \infty} |a_n|^{1/n} < 1$, say $\sum_{n=1}^{\infty} a_n$ converges by the Root Test.
• If \( \lim_{n \to \infty} |a_n|^{1/n} > 1 \) or infinite, say \( \sum_{n=1}^{\infty} a_n \) diverges by the Root Test.

• If \( \lim_{n \to \infty} |a_n|^{1/n} = 1 \), the Root Test is inconclusive. Keep working.

10. Try the Ratio Test. Note that this test works especially well with factorials. Calculate \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \).

• If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \), say \( \sum_{n=1}^{\infty} a_n \) converges by the Ratio Test.

• If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \) or is infinite, say \( \sum_{n=1}^{\infty} a_n \) diverges by the Ratio Test.

• If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \), the Ratio Test is inconclusive. Keep working.

11. If some \( a_n \)'s are negative, you can consider \( \sum_{n=0}^{\infty} |a_n| \).

• If \( \sum_{n=0}^{\infty} |a_n| \) converges, then \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent and therefore convergent. Be sure to state which test you used.

• If \( \sum_{n=0}^{\infty} |a_n| \) diverges, keep working.

12. Try the Integral Test

• Define the corresponding function \( f(x) \).

• Verify that \( f(x) \) is positive on \([1, \infty)\).

• Verify that \( f(x) \) is continuous on \([1, \infty)\).

• Verify that \( f(x) \) is eventually decreasing on \([1, \infty)\).

• Verify that \( \lim_{x \to \infty} f(x) = 0 \).

• If all of these conditions hold, examine \( \int_1^{\infty} f(x) \, dx \).

  – If \( \int_1^{\infty} f(x) \, dx \) converges, say \( \sum_{n=1}^{\infty} a_n \) converges by Integral Test.

  – If \( \int_1^{\infty} f(x) \, dx \) diverges, say \( \sum_{n=1}^{\infty} a_n \) diverges by Integral Test.

• If one of these conditions does not hold, keep working.

13. What do you do if asked to determine whether the series \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent, conditionally convergent, or divergent?

• Are all the \( a_n \)'s positive? If so, absolute convergence and convergence are the same thing for this series, and conditional convergence is not possible. Use an applicable test from above to determine whether the series converges or diverges.
- If \( \sum_{n=1}^{\infty} a_n \) converges, say so and specify which test you used. Then note that since all \( a_n \)'s are positive, \( \sum_{n=1}^{\infty} a_n \) is also absolutely convergent.

- If \( \sum_{n=1}^{\infty} a_n \) diverges, say so and state which test you used.

- If some \( a_n \)'s are negative, consider the series \( \sum_{n=0}^{\infty} |a_n| \). Use any applicable test from above to evaluate this series.
  - If \( \sum_{n=0}^{\infty} |a_n| \) converges, say so and state which test you used. Then say that \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.
  - If \( \sum_{n=1}^{\infty} |a_n| \) diverges, then \( \sum_{n=1}^{\infty} a_n \) must be either conditionally convergent or divergent. Use an applicable test from above to determine whether \( \sum_{n=1}^{\infty} a_n \) converges or diverges.

* If \( \sum_{n=1}^{\infty} a_n \) converges, say it’s conditionally convergent and state which test you used.

* If \( \sum_{n=1}^{\infty} a_n \) diverges, say it’s divergent and state which test you used.