1. (8 points) Find $g'(t)$ given that $g(t) = 6t^5 - 3t^4 + 2t - e^{10}$

Solution:

$$g'(t) = \frac{d}{dt}(g(t)) = \frac{d}{dt}(6t^5 - 3t^4 + 2t - e^{10})$$

$$= \frac{d}{dt}(6t^5) - \frac{d}{dt}(3t^4) + \frac{d}{dt}(2t) - \frac{d}{dt}(e^{10})$$

$$= 6(5t^4) - 3(4t^3) + 2(1), \quad \frac{d}{dt}(e^{10}) = 0 \text{ because } e^{10} \text{ is constant}$$

$$= 30t^4 - 12t^3 + 2$$

2. (8 points) Find $\frac{dv}{dt}$ given that $v = 3t^5 \tan^{-1} t$

Solution: We need to use the product rule here with the fact that $\frac{d}{dt}(t^5) = 5t^4$ and $\frac{d}{dt}(\tan^{-1}(t)) = \frac{1}{1+t^2}$:

$$\frac{dv}{dt} = 3(t^5) \cdot \frac{d}{dt}(\tan^{-1} t) + \tan^{-1} t \cdot \frac{d}{dt}(t^5)$$

$$= 3(t^5) \cdot \frac{1}{1+t^2} + \tan^{-1} t \cdot (5t^4)$$

$$= 3\left(\frac{t^5}{1+t^2} + 5t^4 \tan^{-1} t\right)$$

$$= \frac{3t^5}{1+t^2} + 15t^4 \tan^{-1} t$$

3. (8 points) Find $f'(x)$ given that $f(x) = \frac{x^4+3}{\ln x}$

Solution: We use the quotient rule here, we get

$$f'(x) = \frac{\ln x \cdot \frac{d}{dx}(x^4 + 3) - (x^4 + 3) \cdot \frac{d}{dx}(\ln x)}{(\ln x)^2}$$

$$= \frac{\ln x(4x^3) - (x^4 + 3) \cdot \frac{1}{x}}{\ln^2 x}$$

$$= \frac{4x^4 \ln x - (x^4 + 3)}{x \ln^2 x}$$
4. (8 points) Find $h'(t)$ given that $h(t) = \cos(e^{3t})$

**Solution:** We need to use the chain rule here as follows:

$$h(t) = \cos(e^{3t})$$

$$h'(t) = (-\sin(e^{3t})) \cdot \frac{d}{dt}(e^{3t})$$

$$= -\sin(e^{3t}) \cdot (e^{3t} \cdot \frac{d}{dt}(3t))$$

$$= -\sin(e^{3t})3e^{3t}$$

$$= -3e^{3t} \sin(e^{3t})$$

5. (8 points) Find the slope of the line tangent to the curve $x^3y^2 = 3x + 2y$ at the point $(1, 3)$

**Solution:** We need to use the implicit differentiation to get $y' = \frac{dy}{dx}$:

$$\frac{d}{dx}(x^3y^2) = \frac{d}{dx}(3x + 2y)$$

$$x^3 \cdot \frac{d}{dx}(y^2) + y^2 \cdot \frac{d}{dx}(x^3) = \frac{d}{dx}(3x) + \frac{d}{dx}(2y)$$

$$x^3(2y \frac{dy}{dx}) + y^2(3x^2) = 3 + 2 \frac{dy}{dx}$$

$$2x^3yy' + 3x^2y^2 = 3 + 2y'$$

$$2x^3yy' - 2y' = 3 - 3x^2y^2$$

$$y'(2x^3y - 2) = 3 - 3x^2y^2$$

$$y' = \frac{3 - 3x^2y^2}{2x^3y^2 - 2}$$

Now to find the slope of the line at the point $(1, 3)$ we substitute $x = 1, y = 3$ in the $y'$ above

$$\text{Slope} = \frac{dy}{dx}_{|x=1,y=3} = \frac{3 - 3(1)^2(3)^2}{2(1)^3(3) - 2} = \frac{-24}{4} = -6$$

6. (8 points) A function $f(x)$ has the following second derivative.
\[ f''(x) = (x + 5)^2 - 9 \]

What is the largest open interval upon which the graph of \( f(x) \) is concave down?

**Solution:** We need first to find the inflection point, so first we solve \( f''(x) = 0 \), so we have

\[
\begin{align*}
  f''(x) &= 0 \\
  (x + 5)^2 - 9 &= 0 \\
  (x + 5)^2 &= 9 \\
  x + 5 &= \pm 3 \\
  x &= -2 \text{ or } x = -8
\end{align*}
\]

Now we check the sign of the second derivative before and after the inflection point to see the interval of the concave down:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( (-\infty, -8) )</th>
<th>( (-8,-2) )</th>
<th>( (-2,\infty) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sign of ( f''(x) )</td>
<td>( +++ )</td>
<td>( --- )</td>
<td>( +++ )</td>
</tr>
</tbody>
</table>

we see that \( f''(x) < 0 \) if \(-8 < x < -2\), i.e., the function is concave down when \(-8 < x < -2\).

7. (8 points) A particle moves along the curve given below. As the particle passes through the point \((4, 3)\), its \( x \)-coordinate increases at a rate of 25 cm/s. How fast is the distance from the particle to the origin changing at this instant?

\[ y = \frac{3}{16} x^2 \]

**Solution:** Let \( x \) be the \( x \)-coordinate, \( y \) be the \( y \)-coordinate and \( D \) is the distance from the origin, then we have

Given \( \frac{dx}{dt} = 25 \text{ cm/sec} \)

Required \( \frac{dD}{dt} \) at \( x = 4 \)

Relation \( D = \sqrt{x^2 + y^2} \).

now we need to differentiate the above relation with respect to \( t \), so we get

\[ D = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(\frac{3}{16} y^2\right)^2} = (x^2 + \frac{9}{256} x^4)^{\frac{1}{2}} \]

We need to differentiate the distance function so we use the chain rule to have

\[ \frac{dD}{dt} = \frac{1}{2} (x^2 + \frac{9}{256} x^4)^{-\frac{1}{2}} \cdot (2x \frac{dx}{dt} + \frac{9}{256} (4x^3) \frac{dx}{dt}) \]
Now we substitute in the above equation $x = 4, \frac{dx}{dt} = 15$, we get
\[
\frac{dD}{dt} \bigg|_{(4,3)} = \frac{1}{2}((4)^2 + \frac{9}{256}(3)^4)^{-\frac{1}{2}}(2(4)(25) + \frac{9}{256}(4(4)^3)(25))
\]
\[
= \frac{1}{2}(25)^{-\frac{1}{2}}(200 + 225)
\]
\[
= \frac{425}{10}
\]
\[
= 42.5 \text{ cm / sec}
\]

8. (8 points) A function $g(x)$ has the following derivative.
\[
g'(x) = 6e^x(x - 3)^3(x - 4)^4(x - 7)^6
\]
Determine the $x$-value for each local maximum and local minimum on the graph of $g(x)$.

Solution: We need to find the critical points first and then we use the first derivative test to determine which one is local maximum or local minimum.
\[
g'(x) = 6e^x(x - 3)^3(x - 4)^4(x - 7)^6 = 0 \Rightarrow x = 3, 4, 7
\]
so we have by the first derivative test that

<table>
<thead>
<tr>
<th>$x$</th>
<th>$(-\infty, 3)$</th>
<th>$(3, 4)$</th>
<th>$(4, 7)$</th>
<th>$(7, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sign of $g'(x)$</td>
<td>- - -</td>
<td>+ + +</td>
<td>+ + +</td>
<td>+ + +</td>
</tr>
</tbody>
</table>

we have a local minimum when $x = 3$ and there is no local maximum.

9. (12 points) A ball is tossed straight up with an initial velocity of 16 feet per second. The ball is 4 feet above the ground when it is released. Its height at time $t$ is given by
\[
h = -16t^2 + 16t + 4
\]
What is the ball’s maximum height?

Solution: We need to find the maximum height which will be when the particle has velocity equal to zero and then will drop by the power of the gravity, we have
\[
v(t) = h'(t) = -32t + 16
\]
now we find the critical point, so we have $v(t) = 0 \Rightarrow t_{\text{max}} = 0.5 \text{ sec}$. We need to check that this is a local maximum, so we use the second derivative test (also one can check the first derivative test), we have
\[
h''(t) = -32 \text{ hence } h''(0.5) < 0
\]
Therefore \( t_{\text{max}} \) is indeed a local maximum and so the maximum height is

\[
h(t_{\text{max}}) = h(0.5) = -16(0.5)^2 + 16(0.5) + 4 = -4 + 8 + 4 = 8 \text{ feet.}
\]

10. (8 points each) Evaluate the following limits.

(a) \( \lim_{x \to 2} \frac{x^2 - 5x + 6}{\sin(x - 2)} \)

Solution: By direct substitution, we get \( \frac{0}{0} \) which is undefined, so we apply L’Hopital’s rule, we get

\[
\lim_{x \to 2} \frac{2x - 5}{\cos(x - 2)} = \frac{2(-5)}{\cos(0)} = \frac{-1}{1} = -1
\]

(b) \( \lim_{x \to \pi/4} \frac{4x - \pi}{4 \tan x} \)

By direct substitution, we get \( \frac{0}{0} = 0 \), therefore this is the value of the limit, hence

\[
\lim_{x \to \pi/4} \frac{4x - \pi}{4 \tan x} = \frac{0}{4} = 4
\]

(c) \( \lim_{x \to \infty} \left(1 - \frac{4}{x}\right)^{5x} \)

By direct substitution, we get \( 1^\infty \) which is undeterminant form, now we have

\[
\lim_{x \to \infty} \left(1 - \frac{4}{x}\right)^{5x} = \lim_{x \to \infty} e^{\ln\left(\left(1 - \frac{4}{x}\right)^{5x}\right)}
\]

\[
= \lim_{x \to \infty} e^{5x \ln\left(\left(1 - \frac{4}{x}\right)\right)}
\]

\[
= e^{\lim_{x \to \infty} \left(5x \ln\left(\left(1 - \frac{4}{x}\right)\right)\right)}
\]

(1)

We can exchange the limit and the exponential because the exponential function is continuous and the inside limit is exist as we shall see now, we want to evaluate

\[
\lim_{x \to \infty} \left(5x \ln\left(\left(1 - \frac{4}{x}\right)\right)\right)
\]

which is by direct substitution \( \infty \cdot 0 \) which is indeterminant form and so we need to apply the L’Hopital rule, but here we don’t have a quotient, so we create one and then apply the L’Hopital rule.

\[
\lim_{x \to \infty} \left(5x \ln\left(\left(1 - \frac{4}{x}\right)\right)\right) = \lim_{x \to \infty} \frac{\ln\left(\left(1 - \frac{4}{x}\right)\right)}{\frac{1}{5x}}
\]

\[
= \lim_{x \to \infty} \frac{4}{-5x^2}
\]

\[
= \lim_{x \to \infty} \frac{4 \cdot -5}{\left(1 - \frac{4}{x}\right)} = -20
\]
therefore the limit exist and so we substitute it in (1) to get

\[ \lim_{x \to \infty} \left( 1 - \frac{4}{x} \right)^{5x} = e^{-20} \]

Another way to solve this is by recalling that for any real number \( n \), we have

\[ e^n = \lim_{x \to \infty} \left( 1 + \frac{n}{x} \right)^x \]

so we put \( n = -4 \) we get

\[ e^{-4} = \lim_{x \to \infty} \left( 1 - \frac{4}{x} \right)^x \]

and we take the 5th power of both side and we exchange the limit with the exponential, we get

\[ e^{-20} = \lim_{x \to \infty} \left( 1 - \frac{4}{x} \right)^{5x} \]

\[ \triangleq \]