Introduction to Differential Forms and Connections

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Abstract

In this project, the main aim is to understand the main concepts of differential forms and connection in elementary setting of differential geometry. This is an extra work load for a differential Geometry course that I am taking under Prof. Steven Bradlow at the University of Illinois at Urabana-Champaign. In this course we have to demonstrate the ability to read and understand a topic from the book that is not covered in the lecture, we have to write a short report that shows our understanding and how we can present the topic that we studied by ourself.

The content of my report will be about the basics of differential forms and connections, the report will be 3 chapters, first chapter is about differential form, second about connection and the final chapter is about Connections, especially the Levi Civita connection and the fundamental theorem of Riemannian Geometry in dimension 2).

Main part of the material will be from the text in the course which "Elementary Differential Geometry" by O'Neil. Many of the exercise will be provided as an example to show my understanding to the material and how to present them.

My thanks go to Prof. Bradlow for choosing the material and give me the time and the direction to develop this small report

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Chapter 1 : Differential Forms

§ 1.1 1-Forms:

We start this section by defining 1-Form on the set of all tangent vectors of R^3. So fix a point p ∈ R^3 to be the point of Application.

Definition: (Tangent Space)

The set $T_pR^3 :=\{v_p \mid v \in R^3\}$ is called the tangent space of R^3 at p.

Remark:

1- $T_p R^3$ is an R-vector space with the following addition and scalar multiplication:

$V_p, w_p \in T_p R^3$, $\gamma \in R$,

i) $V_p+w_p:=(v+w)_p$

ii) $\gamma V_p=(\gamma v)_p$

2- $T_p R^3$ is R-isomorphic as an R-vector space to R^3 via the natural map that assign p to the origin point of R^3 i.e. $\varphi: T_pR^3 \ni v_p \to v \in R^3$

Now we define 1-form on R^3 to be an element of the dual space of R^3 i.e.

$(R^3)^* = \{\mu: R^3 \to R \mid \mu \text{ is R-linear}\}$

Definition:

A 1-Form $\{\mu: R^3 \to R \mid \mu$ is R-linear that sends to every tangent vector v of R^3 a real number $\mu(v)$.

Remark:

1) Let $p \in R^3$ be a point of application, then the 1-form ϕ_p is any element of the dual space $(T_pR^3)^*$.

2) Let $F: T_pR^3 \to T_qR^3$ be a map, then $\forall\ \phi_p \in (T_pR^3)^*$, we have

$\phi_q = \phi_p \circ F : T_qR^3 \to R$ is in $(T_qR^3)^*$.

The remaining part of this section, we will be concerned on how to convert a real valued function $f:R^3 \to R$ into a differential 1-Form.

Definition:

Let f: $R^3 \rightarrow R$ be a differentiable map, then the differential $df: T_pR^3 \ni v_p \to grad(f) \cdot v \in R$ is a 1-Form.

Example:

1) Let x_i:$R^n \ni (x_1, x_2, \dots, x_n) \to x_i \in R$ be the i^{th} projection map, then the differential of x_i at any point v_p is $dx_i(v_p)=grad(x_i)=(0,0,\dots,1,\dots,0)v_p=(v_p)_i=i^{th}$ coordinate of v_p.

2) Let f_1,f_2,f_3:$R^3 \rightarrow R$ be a map, then we define a 1-Form

$\psi := f_1dx_1+ f_2dx_2+ f_3dx_3 : T_pR^3 \rightarrow R$ by

$\psi(v) = f_1(p)dx_1(v)+ f_2(p)dx_2(v)+ f_3(p)dx_3(v)$

$= f_1(p)v_1+ f_1(p)v_2+ f_1(p)v_3$

Now one may ask, is every 1-Form can be written in the form in (2) above? i.e. given a 1-form ψ, is there f_1,f_2,f_3 such that $\psi := f_1dx_1+ f_2dx_2+ f_3dx_3$?

The answer is yes, as we shall see in the following lemma.

Lemma 1:

Let μ be a 1-Form on R^3, then $\mu=f_1dX_1+f_2dX_2+f_3dX_3$, where $f_i= \mu(e_i)$.

Proof:

Let $f_i= \mu(e_i)$ (i=1,2,3), then for all $v_p \in T_p R^3$, we have

$(f_1dx_1+f_2dx_2+f_3dx_3)(\text{м}_3)= f_1 (p)dx_1(v_p)+f_2 (p)dx_2(v_p)+f_3 (p)dx_3 (v_p)$

$= f_1 (p)(v_1)+f_2 (p)(v_2)+f_3 (p)(v_3)$

$= \mu(e_1)v_1 + \mu(e_2)v_{2+} \mu(e_3)v_3 = \mu(v_1e_1+ v_2e_2+ v_3e_3)= \mu(v_p)$

So $\mu= f_1dX_1+f_2dX_2+f_3dX_3$ ■

Corollary 2:

Let f be differentiable map on R^3, then

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$$

Proof:

By definition of the differential of f, we have for all $V_p \in T_p R^3$,

$df(v_p)= grad(f)\cdot v=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})\cdot(v_1,v_2,v_3)=\frac{\partial f}{\partial x}v_1+\frac{\partial f}{\partial y}v_2+\frac{\partial f}{\partial z}v_3$, but $v_i=dX_i(v_p)$

So $df(v_p)=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz(v_p)$ ■

Using Corollary 2, we reached to the famous Calculus formula we had in Calc III, here is a list of properties about the differential of a map.

Proposition 3:

Let $f,g:R^3\rightarrow R$, $h:R\rightarrow R$, be differentiable maps, then

i)d(f+g)=d(f)+d(g)

(ii) d(fg)=fdg+gdf

(iii) $d(h\circ f))=h'(f)df$

Proof:

i) $d(f+g)=\frac{\partial(f+g)}{\partial x}dx+\frac{\partial(f+g)}{\partial y}dy+\frac{\partial(f+g)}{\partial z}dz$

$=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz+\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy+\frac{\partial g}{\partial z}dz$

=d(f)+d(g)

ii) put $x_1:=x$, $x_2:=y$, $x_3:=z$, then d(fg)= $\sum_{i=1}^{3}\frac{\partial f}{\partial x}dx$ =$\sum_{i=1}^{3}(\frac{\partial f}{\partial x}g + f\,\frac{\partial g}{\partial x})dx$ =$\sum_{i=1}^{3} g\,\frac{\partial f}{\partial x}\,dx$ + $\sum_{i=1}^{3} f\,\frac{\partial g}{\partial x}dx$=gdf+fdg

iii) h°f:R^3→R, d(h°f)= =$\sum_{i=1}^{3}\frac{\partial h^\circ f}{\partial x}dx$ = =$\sum_{i=1}^{3} h'(f)\frac{\partial f}{\partial x}dx$=h'(f)df ---by chain rule.

■

<u>Example</u>:

1- f: $R^3 \ni (x,y,z) \rightarrow (x^2-1)\, y + (y^2+2)\, z \in R$

df=(2xdx)y+(x^2-1)dy+2y dy z+(y^2+2) dz

= 2xy dx + (x^2-1+2yz) dy + (y^2+2) dz

So as a result, we have $\frac{\partial f}{\partial x}$= 2xy, $\frac{\partial f}{\partial y}$= x^2-1+2yz, $\frac{\partial f}{\partial z}$= y^2+2

2- (exercise 6,b)

f: $R^3 \ni (x,y,z) \rightarrow xe^{yz} \in R$

df= $\frac{\partial f}{\partial x}dx$+$\frac{\partial f}{\partial y}dy$+$\frac{\partial f}{\partial z}dz$= e^{yz} dx+ xz e^{yz} dy + xye^{yz} dz

§1.2- **Differential Form**:

In this section, we will generalize the 1-Form we introduced in section 1, also we might be interesting to work in the n-dimensional Euclidean Space R^n instead of just R^3.

A- Operation on Differentials:

Let dx_i,dx_j be two differentials defined as in section 1, we are interested to define an operation between them called the multiplication of forms and denoted by the symbol $\wedge$ "Wedge operator".

Definition:

Let dx_i,dx_j be two forms, then one define

$dx_i \wedge dx_j := -dx_i \wedge dx_j$ $(i \neq j)$ and 0 if i=j.

Where $dx_i \wedge dx_j$ is an expression for the multiplication $dx_i dx_j$ represents a 2-Form and by the same way we can abstractly define the p-form of R^n .

Definition: (p-Form)

A p-Form is an expression contains p dx_i (i=1,2,…,n) and such an expression is said to have degree p.

Example:

Let f: $R^n \rightarrow R$ be a smooth map, then

1- A 0-Form is just the expression of f.

2- A 1-form is an expression like fdx_1 or fdx_2 ,…

3- A 2-form is an expression like $fdx_i dx_j$

4- A 3-Form is an expression like $fdx_i dx_j dx_k$ and so on…

Remark:

1- In R^n, there is only n-Form, because n+1-Form will contain two similar form dx_i and by wedge rule, we can shift them to be next to each other and then get a result of zero, so if p > n, then p-Forms are automatically zero.

B- Wedge Operator of Forms:

Let ϕ, ψ be two 1-Form in R^n, i.e. $\phi = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} dx_i$ and $\psi = \sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i} dx_i$, then

$\phi \wedge \psi := \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_i} \frac{\partial g_j}{\partial x_j} dx_j dx_i$, So the product of two 1-Form yield 2-form and so in general the product of p-Form and q-Form is p+q-Form.

Example:

1- Let ϕ:=x dx – y dy and ψ:=z dx + x dz , then

$\phi \wedge \psi$ = (x dx – y dy) ∧ (z dx + x dz)

= (xzdxdx+x^2dxdz-zydydx-xydydz

= x^2dxdz-zy dydx-xy dydz

= x^2dxdz+zy dxdy-xydydz --- 2-Form

2- Let ϕ, ψ as above and θ=zdy ,then

$\phi \wedge \psi \wedge \theta$ = (x^2dxdz+zy dxdy-xydydz)∧ (z dy)

= zx^2dxdzdy+z^2 y dxdydy-xyz dydzdy

= zx^2dxdzdy --- 3-Form

3- Let ϕ:=x dx and ψ:=y dxdy , then

$\phi \wedge \psi$ =0

Remark:

Let , ψ be two 1-form, then $\phi \wedge \psi = (\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} dx_i)(\sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i} dx_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_i} \frac{\partial g_j}{\partial x_j} dx_j dx_i =$

$- \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_i} \frac{\partial g_j}{\partial x_j} dx_i dx_j = - \psi \wedge \phi$

Definition: (**Exterior Derivative**)

Let . $\phi = \sum_{i=1}^{n} f_i dx_i$ be the 1-Form on R^n, then the exterior derivative is the 2-form
d $\phi = \sum_{i=1}^{n} df_i \wedge dx_i$

Example:

Let n:=3, then c = $f_1dX_1+f_2dX_2+f_3dX_3$, then

d ϕ= $df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3$

$= (\sum_{i=1}^{3} \frac{\partial f_1}{\partial x_i} dx_i) \wedge dx_1 + (\sum_{i=1}^{3} \frac{\partial f_2}{\partial x_i} dx_i) \wedge dx_2 + (\sum_{i=1}^{3} \frac{\partial f_3}{\partial x_i} dx_i) \wedge dx_3$

$= (\frac{\partial f_1}{\partial x_1} - \frac{\partial f_1}{\partial x_2}) dx_1dx_2 + (\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}) dx_2dx_3 + (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}) dx_3dx_1$

So for example if ϕ = xy dx + x^2 dz, then d ϕ = d(xy) ∧ dx + d(x^2) ∧ dz = (ydx+xdy) ∧ dx + 2x dx ∧ dz = x dydx +2x dxdz

Theorem 1:

Let f be differentiable map, ϕ, ψ are 1-Form, then

i) d(f ϕ)= df ∧ ϕ +f d ϕ

ii) d($\phi \wedge \psi$) = d $\phi \wedge \psi - \phi \wedge$ d ψ

Proof:

i) Let $\phi = \sum_{i=1}^{n} f_i dx_i$, then . $f\phi = \sum_{i=1}^{n} f f_i dx_i$, so

d($f\phi$)= $\sum_{i=1}^{n} f f_i dx_i$)= $\sum_{i=1}^{n} d(f f_i) dx_i$

$= \sum_{i=1}^{n} (d(f) f_i \wedge dx_i + f d(f_i) \wedge dx_i)$

$= \sum_{i=1}^{n} d(f) f_i \wedge dx_i + \sum_{i=1}^{n} f d(f_i) \wedge dx_i$

= df ∧ ϕ +f d ϕ .

ii) Let $\phi = \sum_{i=1}^{n} dx_i$, $= \sum_{i=1}^{n} g_i dx_i$, then

$\phi \wedge \psi = \sum_{i=1}^{n} \sum_{j=1}^{n} f_i g_j dx_j dx_i$, so it is sufficient to prove the statement for $\phi = f dx_i$ and $\psi = g dx_j$ and then extend linearly, so

d($\phi \wedge \psi$)=d($f dx_i \wedge g dx_j$)=d($fg\, dx_i dx_j$)=d(fg) ∧ $dx_i\, dx_j$

$= gd(f)dx_i dx_j + fd(g)dx_i dx_j$

$= d(f)dx_i \wedge gdx_j - dx_i \wedge d(g)dx_j$

$= d(\phi)\wedge\psi - \wedge \phi\, d(\psi)$

■

Example:

1- "ex 2"

Let $\phi = \frac{1}{y}dx$ and $\psi = zdy$, then

$d(\phi\wedge\psi) = d\phi\wedge\psi + \phi\wedge d\psi = d(\frac{1}{y}) \wedge dx\ zdy + \frac{1}{y}\ dx \wedge d(z)\wedge dy$

$= -\frac{1}{y}\frac{1}{y}dy \wedge dxdy + \frac{1}{y}\ dx\frac{1}{y}dz\,\frac{1}{y}dy = \frac{1}{y}dzdxdy$

2- "ex 3"

Let f is a function, then $d(df) = d(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}dx_i) = \sum_{i=1}^{n} d(\frac{\partial f_i}{\partial x_i})\wedge dx_i$

$= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial\partial f_i}{\partial x_j \partial x_i}dx_i dx_j = 0$ and as aresult we have $d(fdg) = df\wedge dg + fd(dg) = df\wedge dg$

3- differential Form in Cylindrical Coordinates:

Let $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, then

$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta$

$dy = \sin\theta dr + r\cos\theta d\theta$

$dz = dz$

So for example, dxdydz = volume element = $(\cos\theta dr - r\sin\theta d\theta) \wedge (\cos\theta dr - r\sin\theta d\theta) \wedge dz = rdrd\theta\, dz$

4- Let $f,g : R^2 \rightarrow R$, then

$df \wedge dg = (f_x dx + f_y dy) \wedge (g_x dx + g_y dy)$

$= (f_x g_y dxdy + f_y g_x dydx) = (f_x g_y dxdy - f_y f_x)dxdy$

$= \det \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ dxdy --- as we have in calculus

§ 1.3: **Differential forms on a surface**:

In section 1.2, we discussed the differential n-form in R^n, in this section we will be interesting on forms on a surface $M \subseteq R^3$, for 0-Form & 1-Form, we keep the same definition as in the previous section, rather we will give an abstract definition for the 2-Form on a surface and we will show it coincide with the definition in Section 1.2.

Definition:

A 2-Form μ on a surface $M \subseteq R^3$ is a map $\mu: T_pM \rightarrow R$ such that for all $v,w \in R^3$

tangents to M at p, we have

i) $\mu(v,w)$ is R-bilinear.

ii) $\mu(v,w)=-\mu(w,v)$

Remark:

1- Since M is two-dimensional space, then all p-form are zero for p>2.

2- m(v,v)=- m(v,v) , then m(v,v)=0

3- Let {v,w} be R-linearly tangent vectors to M at p, then

$$M(av+bw,cv+dw)=a\,\mu(v,cv+dw)+b\,\mu(w,cv+dw)$$

$$=a[\,c\,\mu(v,v)+d\,\mu(v,w)]+b[\,c\,\mu(w,v)+d\,\mu(w,w)\,]$$

$$=ad\,\mu(v,w)+bc\,\mu(w,v)=(ad-bc)\,\mu(v,w)=\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mu(v,w)$$

So by remark 3, we can easily compute the 2-form for a given R-basis for T_pM on M.

Definition:

Let ϕ,ψ be 1-form on a surface, then the wedge product is a 2-form on M such that $(\phi\wedge\psi)(v,w)=\phi(v)\,\psi(w)-\phi(w)\,\psi(v)$, for all v,w are tangent to M.

Note:

$\phi\wedge\psi = -\psi\wedge\phi$ and $\phi\wedge\phi = 0$, hence all algebraic properties established in section 1.2 will be applicable here also, now we have to give an abstract meaning to the exterior derivative of a 2-form.

Definition: (Exterior Derivative)

Let ϕ be 1-Form on a surface M, then the <u>exterior derivative</u> $d\phi$ of ϕ is a 2-form such that for all patch σ on M, we have

$$d\,\phi\,(\sigma_s, \sigma_t) := \frac{\partial}{\partial s}(\phi(\sigma_t)) - \frac{\partial}{\partial t}(\phi(\sigma_s))$$

Now before continue developing the theorems about the exterior derivative, we have to show that our definition is well-defined i.e. doesn't depend only on the chosen patch σ.

Lemma 1:

Let ϕ be 1-form on M. Let $\sigma: D\subseteq R^2 \rightarrow R^3$, $\pi: E\subseteq R^2 \rightarrow R^3$ be two patches on M,

then $d_\sigma\,\phi = d_\pi\,\phi$ on the overlap of $\sigma(D)$, $\pi(E)$

Proof:

Since π_u, π_v are linearly independent, it sufficient to show that $d_\pi\,\phi(\pi_u, \pi_v)$

$= d_\sigma\,\phi\,(\pi_u, \pi_v)$

Now let $\pi = \sigma(u', v')$, where by chain rule we have

$$\pi_u = \frac{\partial u'}{\partial u}\sigma_u + \frac{\partial u'}{\partial u}\sigma_v \quad \text{and} \quad \pi_v = \frac{\partial u'}{\partial v}\sigma_u + \frac{\partial v'}{\partial v}\sigma_v$$

so $d_\sigma\,\phi\,(\pi_u, \pi_v) = J(d_\sigma\,\phi)(\sigma_u, \sigma_v)$ and so we want to show

$$\frac{\partial}{\partial u}\phi(\pi_v) - \frac{\partial}{\partial v}\phi(\pi_u) = J\left\{\frac{\partial}{\partial u}\phi(\sigma_v) - \frac{\partial}{\partial u}\phi(\sigma_v)\right\}$$

Now $\phi(\pi_u) = \frac{\partial u'}{\partial u}\phi\,(\sigma_u) + \frac{\partial u'}{\partial u}\phi(\sigma_v)$\

$$\frac{\partial}{\partial u}\phi(\pi_u) = \frac{\partial}{\partial u}\phi(\sigma_u)\frac{\partial u'}{\partial v} + \frac{\partial}{\partial u}\phi(\sigma_v)\frac{\partial v'}{\partial v} + \ldots$$, where .. means mixed derivatives

Also

$-\frac{\partial}{\partial v}\phi(\pi_v) = -\frac{\partial}{\partial v}\phi(\sigma_u)\frac{\partial u'}{\partial v} - \frac{\partial}{\partial v}\phi(\sigma_v)\frac{\partial v'}{\partial v} \quad + \ldots.$

$\frac{\partial}{\partial u}\phi(\sigma_v) - \frac{\partial}{\partial u}\phi(\sigma_v) = J(u',v')\{\frac{\partial}{\partial u'}\phi(\sigma_v) - \frac{\partial}{\partial u'}\phi(\sigma_v)\}$ ■

Theorem 2:

If f: $M \subseteq R^3 \rightarrow R$ is differentiable map, then d(df)=0

Proof:

Let ψ :=df, let be any patch on M, we want to show that $d(\psi)=0$ i.e. $d\,\psi(\sigma_s, \sigma_t)=0$

$\Psi(\sigma_s) = df(\sigma_s) = \frac{\partial f}{\partial s}(\sigma)$ and $\Psi(\sigma_t) = df(\sigma_t) = \frac{\partial f}{\partial t}(\sigma)$

So $d\psi(\sigma_s, \sigma_t) = \frac{\partial}{\partial s}(\Psi(\sigma_t)) - \frac{\partial}{\partial t}(\Psi(\sigma_s))$

$= \frac{\partial}{\partial s}(\frac{\partial f}{\partial t}(\sigma)) - \frac{\partial}{\partial t}(\frac{\partial f}{\partial s}(\sigma)) = \frac{\partial\partial f}{\partial s\partial t} - \frac{\partial\partial f}{\partial t\partial s} = 0$

■

Note:

1- In theory of surfaces, whenever the exterior derivative applied twice to a form, it gives always zero, if we start with 0-form, theorem 2 tell us d(df)=0, if we pick a 1-form, then the twice derivative gives us 3-form which is by definition zero.

2- All operation and machinery developed on § 1.2 can be carried here e.g. if f,g & h are differentiable maps on M, then

i)d(fgh)=ghd(f) + fhd(g)+fgd(h)

ii) $d(f\phi) = fd\phi - \phi \wedge df$

iii) $d(f \wedge dg)(v,w) = \nabla_v f\, \nabla_w g - \nabla_v g \nabla_w f$

We conclude this section, by the following definition,

Definition:

1- A differential form ϕ is called <u>closed</u> if $d\phi=0$

2- A differential form ϕ is called <u>exact</u> if $\exists\ \xi$ form such that $\phi = d\,\xi$

Note:

1- exact $\Longrightarrow$ closed (theorem 2)

2- An exact form reminds me by the primitive (anti-derivative) function on the Calculus.

Chapter 2: Connection form

§ 2.1 Covariant Derivative:

In this section we will be interested to find a notion of the derivative in any vector field W: $R^3 \rightarrow R^3$, of course we will use both definition of analysis (Jacobi-Matrix) and the differential geometry (using derivative of a curve)

Definition:

Let F: $R^3 \rightarrow R^3$ be a vector field, let $v_p \in T_pR3$, then the covariant derivative of F with respect to v_p is

$\nabla_v F := \frac{d}{dt} F(p+tv)(0)$

Remark:

Let e_1, e_2, e_3 be the standard basis for T_pR^3 , then

$\nabla_v F = F' \cdot v$, where the multiplication is matrix multiplication.

Proof of the remark:

Let $F = (f_1, f_2, f_3) : R^3 \rightarrow R^3$, then $F(p+tv) = (f_1(p+tv), f_2(p+tv), f_3(p+tv))$

So $F(p+tv)' = (f_1'(p+tv), f_2'(p+tv), f_3'(p+tv)) = (\nabla_v f_1, \nabla_v f_2, \nabla_v f_3) = F' \cdot v$ ■

Example:

1- Let $F: R^3 \ni (x,y,z) \rightarrow (x^2,0,yz) \in R^3$, $v := (-1,0,2)$, $p := (2,1,0)$, then

$$\nabla_v F = F' \cdot v = \begin{pmatrix} 2x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z & y \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -x \\ 0 \\ 2y \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}$$

2- Ex.1

Let $F: R^3 \ni (x,y,z) \rightarrow (x^2,y,0) \in R^3$, $v := (1,-1,2)$, $p := (1,3,-1)$, then

$$\nabla_v F = F'\cdot v = \begin{pmatrix} 2x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2x \\ -1 \\ 0 \end{pmatrix} = \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

3-

Let $F:R^3 \ni (x,y,z) \rightarrow (x,x^2,-z^2) \in R^3$, $v := (1,-1,2)$, $p := (1,3,-1)$, then

$$\nabla_v F = F'\cdot v = \begin{pmatrix} 1 & 0 & 0 \\ 2x & 0 & 0 \\ 0 & 0 & -2z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2x \\ -4z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

And by the previous remark, we can prove the following corollary:

Corollary 1: (Linear and Leibnizian properties)

Let $v,w \in T_pR3$, let $F,G:R^3 \rightarrow R^3$, then for $a,b \in R$, $f:R^3 \rightarrow R$

(1) $\nabla_{av+bw}F = a\nabla_v F + b\nabla_w F$

(2) $\nabla_v(aF+bG) = a\nabla_v F + b\nabla_v G$

(3) $\nabla_v(f\,F) = \nabla_v f\, F(p) + f(p)\, \nabla_v F$

(4) $\nabla_v(F\,G) = \nabla_v F\, G + G\, \nabla_v F$

Proof:

Only (3) and (4) need to be proven,

(3) $\nabla_v(f\,F) = (f\,F)'(p+tv) = f'(p+tv)\cdot F(p+tv) + f(p+tv)\cdot F'(p+tv)$ at $t=0$

so $\nabla_v(f\,f) = \nabla_v f\, F(p) + f(p)\, \nabla_v F$

(4) Let $F=(f_1,f_2,f_3)$, $G=(g_1,g_2,g_3)$, then $F\cdot G = f_1g_1+f_2g_2+f_3g_3$ and so

$\nabla_v(F\cdot G) = \nabla_v(f_1g_1+f_2g_2+f_3g_3) = \nabla_v(f_1g_1) + \nabla_v(f_2g_2) + \nabla_v(f_3g_3)$

$= \nabla_v(f_1)g_1 + \nabla_v(g_1)f_1 + \nabla_v(f_2)g_2 + \nabla_v(g_2)f_2 + \nabla_v(f_3)g_3 + \nabla_v(g_3)f_3$

$= \nabla_v(F)\cdot G + \nabla_v(G)\cdot F$ ■

Example:

Let $F: R^3 \rightarrow R^3$ be a vector field with $||F||$=constant, then for any $v \in T_pR^3$, we have $\nabla_v(||F||)=0$, then $\nabla_v(F\cdot F) = \nabla_v F\cdot F = 0$ i.e. the covariant derivative is orthogonal to F.

Definition: (Frame Field)

Vectors $E_1,E_2,E_3 \in R^3$ are called Frame Field if $\{ E_1,E_2,E_3 \}$ is orthogonal R-basis for R^3.

Example:

1- Let r,θ,z be the coordinate system for the cylindrical coordinates of R^3 ,then

$E_1=(\cos \theta ,\sin \theta , 0)$

$E_2 = (-\sin \theta ,\cos \theta ,0)$

$E_3= (0,0,1)$

2- Let v,w be two R-linearly independent vector field in R^3 , then we want to find frame field from them, then put $E_1:=F/\|F\|$, now put

$E_2=(w-w\cdot E_1)E_1 \setminus \|(w-w\cdot E_1)E_1\|$ and so we choose $E_3=E_1 \times E_2 / \| E_1 \times E_2 \|$

§ 2.2 <u>Connection Forms:</u>

<u>Recall</u>:

Given a smooth curve $\alpha : [0,1] \subseteq R \rightarrow R^3$, we define a frame field called "Frenet Frame" as follows, we reparametrize the curve to get unit-speed curve $\beta : [a,b] \subseteq R \rightarrow R^3$, then

$T(t) := \beta'(t)$

$N(t) := T'(t) \backslash \|T'(t)\|$ (provided $k(t) := \|T'(t)\| \neq 0$

$B(t) := T(t) \times N(t)$

And then we studied in details the variation of this frame and we get this matrix notation:

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & t \\ 0 & t & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

In this section, we will try to generalize this result to any <u>frame</u> field $\{ E_1,E_2,E_3 \}$ of R^3 and we want to study the variation problem, i.e. express $\nabla_v E_1$, $\nabla_v E_2$, $\nabla_v E_3$ (v is vector field) in term of E_1,E_2,E_3 .

<u>Construction</u>:

Fix $p \in R^3$ and $v \in T_pR^3$ let E_1,E_2,E_3 be a frame field for T_pR^3, then $\nabla_v E_1$, $\nabla_v E_2$, $\nabla_v E_3 \in T_pR^3 = \text{span}_R \{E_1,E_2,E_3\}$

So

$\nabla_v E_1 = c_{11}E_1 + c_{12}E_2 + c_{13}E_3$

$\nabla_v E_2 = c_{21}E_1 + c_{22}E_2 + c_{23}E_3$

$\nabla_v E_3 = c_{31}E_1 + c_{32}E_2 + c_{33}E_3$

Where $c_{ij} = \nabla_v E_i \cdot E_j$ $(1 \leq i,j \leq 3)$

<u>Definition</u>:

The coefficients c_{ij} are called the connection form and denote by

$c_{ij} =: \omega_{ij}(v) = \nabla_v E_i \cdot E_j$ $(1 \leq i,j \leq 3)$

<u>Lemma 1</u>:

The connection form are 1-form.

Proof:

It is enough to show that $\omega_{ij} \in (T_pR^3)^*$, i.e. $w_{ij}: T_pR^3 \rightarrow R$ is R-linear map, so let $v,w \in T_pR^3$, a,b $\in$ R, then

$$\omega_{ij}(av+bw) = \nabla_{av+bw}E_i \cdot E_j = (a\nabla_v E_i + b\nabla_w E_i)\cdot E_j = a\nabla_v E_i \cdot E_j + b\nabla_w E_i \cdot E_j$$

$$= a\omega_{ij}(v) + b\omega_{ij}(w) \qquad \blacksquare$$

Remark:

1- $\omega_{ij}(v)=-\omega_{ji}(v)$ $(1\leq i,j\leq 3)$

2- The computation of the connection is reduced only to compute 3 out of the 9 connections, i.e. we need only to find $\omega_{12}(v)$, $\omega_{13}(v)$, $\omega_{23}(v)$.

Proof of the remark 2:

We have $E_i\cdot E_j = \delta_{ij}$ since E_i,E_j are taken to be orthonormal and δ_{ij} is the kronecker symbol, then $\nabla_v E_i \cdot E_j + E_i\cdot\nabla_v E_j = 0$, so $\omega_{ij}(v) + \omega_{ji}(v) = 0$ and so $\omega_{ij}(v)=-\omega_{ji}(v)$ and also $\omega_{ii}(v) = 0$, for all i=1,2,3

3- For any vector field $V:R^3\rightarrow R^3$,we have

$$\nabla_v E_i = \omega_{i1}(v) \cdot E_1 + \omega_{i2}(v) \cdot E_2 + \omega_{i3}(v) \cdot E_3$$

Notation:

By the above remark, by finding $\omega_{12}(v)$, $\omega_{13}(v)$, $\omega_{23}(v)$, one can use this matrix

$$W = \begin{matrix} 0 & \omega_{12} & \omega_{12} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{matrix}$$ Which is skew symmetric matrix

and so we have the following:

$\nabla_v E_1 = \quad \omega_{12}(v) \cdot E_2 + \omega_{13}(v) \cdot E_3$

$\nabla_v E_2 = -\omega_{12}(v) \cdot E_1 + \quad + \omega_{23}(v) \cdot E_3$

$\nabla_v E_3 = -\omega_{13}(v) \cdot E_1 - \omega_{23}(v) \cdot E_2$

Lemma: (exercise 8) “Showing that the Frenet Frame is special case of the connection equation”

Let β be a unit speed curve with $\kappa>0$, let $E_1=T(t)$, $E_2=N(t)$,$E_3=B(t)$ be the Frenet Frame, then $\omega_{12}(T)= \kappa$, $\omega_{13}(T)=0$ and $\omega_{23}(T)= \tau$

Proof:

By definition, $\omega_{12}(T) = \nabla_v E_1 \cdot E_2 = \nabla_T T \cdot N = T' \cdot N = \kappa\, N \cdot N = \kappa$

$\omega_{13}(T) = \nabla_v E_1 \cdot E_3 = \nabla_T T \cdot B = T' \cdot B = \kappa\, N \cdot B = 0$
$\omega_{23}(T) = \nabla_v E_2 \cdot E_3 = \nabla_T N \cdot B = \tau\, B \cdot B = \tau$ ■

Now let E_1,E_2,E_3 be a frame field for R^3 , we express it in term of the natural basis $\{e_1,e_2,e_3\}$ of R^3 and we get the transition matrix $A:=(a_{ij})$, where

$E_1=a_{11}e_1+ a_{12}e_3+ a_{13}e_3$

$E_2=a_{21}e_1+ a_{22}e_3+ a_{23}e_3$ again $a_{ij}=E_i \cdot e_j$

$E_3=a_{31}e_1+ a_{32}e_3+ a_{33}e_3$

Now we define $dA := (da_{ij})$, the <u>attitude matrix</u> of 1-Form of A, then we have the following important theorem.

Theorem 2:

Let $A=(a_{ij})$ as above, let $\omega := (\omega_{ij})$ be the connection form of E_1, E_2, E_3 , then

$\omega = dA \cdot A^T$ (matrix multiplication)

Proof:

We need to show that ∀ $1 \le i,j \le 3$, we have

$\omega_{ij} = a_{j1}\, da_{i1} + a_{j2}\, da_{i2} + a_{j3}\, da_{i3}$.

Since ω_{ij} is 1-Form i.e. R-linear map, it is enough to show equality for any vector v T_pR^3 , so

$\omega_{ij}(v) = \nabla_v E_i \cdot E_j$

$= \nabla_v E_i \cdot (a_{j1}e_1 + a_{j2}e_3 + a_{j3}e_3)$

$= \nabla_v(a_{i1}e_1 + a_{i2}e_3 + a_{i3}e_3) \cdot (a_{j1}e_1 + a_{j2}e_3 + a_{j3}e_3)$

$= \nabla_v a_{i1}e_1 \cdot a_{j1}e_1 + \nabla_v a_{i2}e_2 \cdot a_{j2}e_2 + \nabla_v a_{i3}e_3 \cdot a_{j3}e_3$ ---because of othogonality $= (a_{j1}\nabla_v a_{i1} + a_{j2}\nabla_v a_{i2} + a_{j3}\nabla_v a_{i3})(v)$

$= (a_{j1}\, da_{i1} + a_{j2}\, da_{i2} + a_{j3}\, da_{i3})\,(v)$ ■

So by having the attitude matrix, one can easily find the connection forms, we conclude this section by the following example

Example: (ex.3)

Let $f : R^3 \rightarrow R$ be any differentiable map, let $A := \begin{matrix} cos^2 f & cosf sinf & sinf \\ sinf cosf & sin^2 f & -cosf \\ -sinf & cosf & 0 \end{matrix}$

Direct computations, show that this is indeed an attitude matrix because it has orthogonal row vectors, now

$$\omega = dA \cdot A^T = \begin{matrix} -2cosf sinf df & (-sin^2 f + cos^2 f)df & cosf\ df \\ (cos^2 f - sin^2 f)df & 2sinf cosf\ df & sinf df \\ -sinf\ df & -sinf\ df & 0 \end{matrix} \quad (A^T)$$

So $\omega_{ij} = a_{j1}\, da_{i1} + a_{j2}\, da_{i2} + a_{j3}\, da_{i3}$

$\omega_{12} = a_{21}\, da_{12} + a_{22}\, da_{12} + a_{23}\, da_{12}$

$= (\sin f \cos f)(-2 \cos f \sin f) df + \sin^2 f\,(-\sin^2 f + \cos^2 f)\, df - \cos^2 f\, df$

$= (-\sin^2 f \cos^2 f - \sin^4 f + \sin^2 f \cos^2 f - \cos^2 f)\, df$

$= (-\sin^2 f \cos^2 f - \sin^4 f - \cos^2 f)\, df$

$= (-\sin^2 f - \cos^2 f)\, df = -df$

By the same way we will have the following:

$\omega_{13} = \cos f\, df \qquad \omega_{23} = \sin f\, df$

$$\omega = \begin{matrix} 0 & -df & \cos f\, df \\ df & 0 & \sin f\, df \\ -\cos f\, df & -\sin f\, df & 0 \end{matrix}$$

§ 2.3 **The Structural Equations:**

Definition: (Dual 1-form)

Let E_1,E_2,E_3 be a frame field of R^3, the dual 1-form are $\theta_1, \theta_2, \theta_3 : R^3 \rightarrow R$ such that

$\theta_i(v) = v \cdot E_i$

Remark:

1- If e_1,e_2,e_3 are the natural R-basis, then $\theta_i(v)=v\cdot e_i = v_i=dx_i(v)$, so $\theta_i=dx_i$

2- $\theta_i (E_j) = E_j \cdot E_i = \delta_{ij}$

3- Let ϕ be a one form of R^3, then $\phi = \phi (E_1)\theta_1 + \phi (E_2)\theta_2 + \phi (E_3)\theta_3$ because for any $v \in R^3$, then

4-Let A be the attitude matrix, then $\theta_i= a_{i1}\, dx_1 + a_{i2}\, dx_2 + a_{i3}\, dx_3$ so again by the attitude matrix we can compute both the connection forms and the dual forms.

Theorem 1: (Cartan Structural Equation)

Let E_1,E_2,E_3 be a frame field on R^3 with the dual forms $\theta_1, \theta_2, \theta_3$ and connection forms ω_{ij} $(1\leq i,j\leq 3)$, then

(1) The First Structural Equation

$$d\theta_i= \sum_{j=1}^{3} \omega ij \wedge \theta i$$

(2) The Second Structural Equation

$$d\,\omega_{ij}= \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj}$$

Proof:

1) Let $\theta := (\theta_1, \theta_2, \theta_3)^T$, $\xi := (dx_1,dx_2,dx_3)^T$ and A is the attitude matrix where $A^TA=I_3$ (A is orthogonal matrix) now $\theta = A\, d\,\xi$, so $d(\theta) = d(Ad\,\xi)=dA \cdot d\,\xi + d^2\,\xi$

$d\,\theta =dA \cdot d\,\xi = dA\cdot A^{T\cdot}Ad\,\xi = \omega\,\theta$, where $\omega\,\theta$ means multiplication by wedge product and hence we got the first structural equation

2) $d\omega = d(dA \cdot A^T) = d(A\, dA^T)^T = -dA \wedge d(A^T) = -dA \cdot A^T \cdot A(dA)^T = -\,\omega \wedge \omega = \omega^2$ ■

We conclude this section by an example illustrate the Cartan structural equation.

Example:

Let E_1, E_2, E_3 be the frame field of the spherical coordinates of R^3, then

$\theta_1 = dr$ $\omega_{12} = \cos\varphi d\theta$ $x_1 = r\cos\varphi \cos\theta$

$\theta_2 = r\cos\varphi d\theta$ $\omega_{13} = d\varphi$ $x_2 = r\cos\varphi \sin\theta$

$\theta_3 = rd\varphi$ $\omega_{23} = \sin\varphi d\theta$ $x_3 = r\sin\theta$

By the first structural equation, we have

$d\theta_3 = \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 + \omega_{33} \wedge \theta_3 = \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2$

$= -d\varphi \wedge dr - \sin\varphi d\theta \wedge r\cos\varphi d\theta$

$= dr \wedge d\varphi$

By the second structural equation

$d\omega_{12} = \omega_{11} \wedge \omega_{12} + \omega_{12} \wedge \omega_{22} + \omega_{13} \wedge \omega_{32} = d\varphi \wedge -\sin\varphi d\theta = \sin\varphi \, d\theta d\varphi$

§ 2.4: The Fundamental Equation of a surface in R^3:

In § 2.3, we studied the frame field of R^3 and develop theories about the 1-form, connection forms and the dual 1-forms. In this section we are interesting to study and apply Cartan's method on a surface $M \subseteq R^3$, so we begin to define a frame field on a surface as follows:

Definition: (Adapted Frame Field)

An Adapted Frame Field E_1,E_2,E_3 in a region D of $M \subseteq R^3$ is an Euclidean frame field such that E_3 is always normal to M and so $E_1,E_2 \in T_p(M)$

So the normal vector field n(t) we used to, we have just to replace it by E_3.

Lemma 1:

There is an adapted frame field on a region $D \subseteq M$ if and only if D is orientable and there exist a nonvanishing tangent vector field on D.

Proof:

$\Longrightarrow$) Let E_1,E_2,E_3 be an adapted frame field, then E_3 is normal to M, so E_3 orients D, now $E_1,E_2 \in T_p(M)$, so there exist tangent vector fields on d which is nonvanishing.

$\Longleftarrow$) Let n be the unit normal vector such that n orients D, let $V:M \rightarrow R^3$ be a vector field doesn't vanish on D, then $E_1 := \frac{V}{||v||}$, $E_2 := n \times E_1$, E_3:=n which is an adapted frame field.

■

Example:

1- Let $M := \{(x,y,z) \in R^3 \mid x^2+y^2=r^2\}$ be the cylinder surface, let

$E_3 := \frac{grad(g)}{||grad(g)||} = \frac{(2x,2y,0)}{\sqrt{4x^2+4y^2}} = (x,y,0)/r$

E_1:= k is tangent to M and so $E_2 := E_1 \times E_3 = (-y,x,0)/r$

And thus we have an adapted frame field on the whole region of the cylinder.

2- $S^2_r := \{ (x,y,z) \in R^3 \mid x^2+y^2+z^2=r^2\}$

$D := S^2_r \setminus \{NP,SP\}$ $\quad$ $E_3 := (x,y,z)\backslash r$

Let V: $R^3 \ni (x,y,z) \rightarrow (-y,x,0) \in R^3$ which vanishes only in the north and south pole (0,0,+-r) which is not part of D,so V doesn't vanish at D and so $E_1 := V \setminus \|V\|$ $E_2 := E_1 \times E_3$

And hence we got an adapted frame field for $D := S^2_r \setminus \{NP,SP\}$

1-**Form Connections of a surface M**:

Now let E_1,E_2,E_3 be an adapted frame field on a surface M, let $v_p \in T_pM$, then by the same calculation in § 2.2, we have

$\nabla_v E_i := \omega_{i1}(v) \cdot E_1 + \omega_{i2}(v) \cdot E_2 + \omega_{i3}(v) \cdot E_3$ and in particular the connection becomes a 1-form on the surface M.

Corollary 2:

Let S be the shape operator on M gotten by the normal vector E_3, then for all $v_p \in T_pM$, we have $S(v) = \omega_{13}(v) \cdot E_1 + \omega_{23}(v) \cdot E_2$

Proof:

$$S(v) := -\nabla_v n = -\nabla_v E_3 = -(\omega_{31}(v) \cdot E_1 + \omega_{32}(v) \cdot E_2 + \omega_{33}(v) \cdot E_3)$$

$$= \omega_{13}(v) \cdot E_1 + \omega_{23}(v) \cdot E_2$$ ■

2- **1-Form on a surface M**:

Let E_1,E_2,E_3 be an adapted frame field, then the one form is defined

$\theta_i: T_pM \ni v \rightarrow v \cdot E_i \in T_pM$

Now $\theta_3(v) = v \cdot E_3 = 0$, so θ_3 is always zero on the surface (because M is 2 dimensional)

Example:

Let the spherical coordinates as in example 1, b, we have the adapted frame field

$E_1 = V \setminus \|V\| = (-y,x,0) \setminus r$ $E_3 = (x,y,z) \setminus r$ $E_2 = E_3 \times E_1$

So $\theta_1 = r\cos\varphi\, d\theta$ $\omega_{12} = \sin\varphi\, d\theta$

$\theta_2 = r d\varphi$ $\omega_{23} = -\cos\varphi\, d\theta$ See § 2.3

$\theta_3 = 0$ $\omega_{23} = -\,d\varphi$

<u>Theorem 3 (the fundamental Equations on a surface)</u>:

If E_1, E_2, E_3 is any adpted frame field on a surface $M \in R^3$, then

(1) $d\theta_1 = \omega_{12} \wedge \theta_2$ --- First Structural Equation

$d\theta_2 = \omega_{21} \wedge \theta_1$

(2) $\omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 = 0$ --- Symmetry Equation

(3) $d\omega_{21} = \omega_{13} \wedge \omega_{32}$ --- Gauss Equation

(4) $d\omega_{13} = \omega_{12} \wedge \omega_{23}$ --- Codazzi Equation

$d\omega_{23} = \omega_{21} \wedge \omega_{13}$

<u>Proof</u>:

(1) by the cartan's first structural equation,

$d\,\theta_1 = \omega_{12} \wedge \theta_2 + \omega_{12} \wedge \theta_2\ \omega_{12} \wedge \theta_2 = \omega_{12} \wedge \theta_2$

(2) $d\,\theta_3 = \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 = 0$

(3) by Cartan's second structural formula,

$d\omega_{13} = \omega_{11} \wedge \omega_{12} + \omega_{12} \wedge \omega_{22} + \omega_{13} \wedge \omega_{32} = \omega_{13} \wedge \omega_{32}$

■

§ 2.5: **<u>Form Computation</u>**:

Let E_1,E_2,E_3 be an adapted frame field on a surface $M \subseteq R^3$, then we have $T_pM = span_R\{ E_1,E_{,2} \}$, so to say that two 1-form ϕ,ψ are equal it is enough to show that $\phi(E_i)= \psi (E_i)$ (i=1,2) and also for 2-form on a surface i.e.

$\mu = \upsilon \Leftrightarrow \mu (E_1,E_2) = \upsilon (E_1,E_2)$

<u>Lemma 1</u>:

Let θ_1, θ_2 be the dual 1-form of E_1,E_2 on the surface M, let ϕ be 1-form, μ be 2-form, then

1) $\phi= \phi(E_1)\, \theta_1+ \phi(E_2)\, \theta_2$

2) $\mu = \mu(E_1,E_2)\, \theta_1 \wedge \theta_2$

<u>Proof</u>:

1)let $v \in T_pM$, $\phi(E_1)\, \theta_1+ \phi(E_2)\, \theta_2 (v)= \phi\, (\theta_1 (v)E_1 + \theta_2 (v)E_2) =\phi(v)$

2) $\mu(E_1,E_2)\, \theta_1\wedge \theta_2 (E_1,E_2) = \mu(E_1,E_2)[\theta_1\wedge \theta_2(E_1,E_2)]$

$$= \mu(E_1,E_2) [\theta_1(E_1)\, \theta_2(E_2)- \theta_1(E_2)\theta_2(e_1)]$$

$$= \mu(E_1,E_2) [1]= \mu(E_1,E_2)$$ ■

<u>Lemma 2</u>:

Let K be the gauss curvature, H be the mean curvature, then

(1) $\omega_{12} \wedge \omega_{23} = K\, \theta_1\wedge\theta_2$

(2) $\omega_{13} \wedge \theta_2 + \theta_2 \wedge\omega 12 = 2H\, \theta_1\wedge\theta_2$

<u>Proof</u>:

By Corollary 2, §2.4, now for all $v \in T_pM$, we have

$S(v) = \omega_{13}(v)E_1 + \omega_{23}(v)E_2$

$S(E_1) = \omega_{13}(E_1)E_1 + \omega_{13}(E_1)E_2$

$S(E_2) = \omega_{13}(E_2)E_1 +\omega_{13}(E_2)E_2$

So by linear algebra, we have S as matrix

$$S=\begin{pmatrix}\omega_{13}(E_1) & \omega_{13}(E_2)\\ \omega_{23}(E_1) & \omega_{23}(E_2)\end{pmatrix}$$

Recall that K=det(S) and H=1\2 Trace(S) so

K= det(S)= $\omega_{13}(E_1)\,\omega_{23}(E_2)$- $\omega_{13}(E_2)\,\omega_{23}(E_1)$ = $(\omega_{12} \wedge \omega_{23})\,(E_1,E_2)$

Now by lemma 1, $\omega_{12} \wedge \omega_{23} = K\,\theta_1\wedge\theta_2$

Also H=1\2 trace(S)=1\2 $(\omega_{13}(E_1) + \omega_{23}(E_2))$

$\omega_{13}(E_1) + \omega_{23}(E_2)= 2H \Rightarrow (\omega_{13} \wedge \theta_2 + \theta_2 \wedge\omega 12)(E_1,E_2)=2H$

$\omega_{13} \wedge \theta_2 + \theta_2 \wedge\omega 12 = 2H\,\theta_1\wedge\theta_2$ ■

Corollary 3:

$d\,\omega_{12} = K\,\theta_1\wedge\theta_2$

Proof:

Gauss equation of theorem 3, §2.4 says that $d\,\omega_{12} = \omega_{13} \wedge \omega_{32}$, by lemma 2 $d\,\omega_{12} = \omega_{12} \wedge \omega_{32} = K\,\theta_1\wedge\theta_2$ ■

Definition: (principle Frame)

A principle frame field on M R^3 is an adapted frame field E_1,E_2,E_3 such that E_1 & E_2 are principle tangent vectors on M.

Now the following lemma, tells us that we always can find a local principle frame field.

Lemma 4: (Existence of principle frame)

Let p M be nonumbilic point, then there exist a principle frame field on some neighborhood of p in M.

Proof:

Let F_1,F_2,F_3 be arbitrary adapted frame field on some neighborhood N of p, since p is not umbillic, then $k_1(p)\neq k_2(p)$, now let $S := (S_{ij})$ be the shape operator matrix with respect to F_1,F_2 , now

$V_1 := S_{12}F_1 + (k_1-S_{11}) F_2$

$V_2 := (k_2-S_{22})F_1 + S_{12}F_2$

Now these vectors are the candidate to the frame field to be principle.

Now (V_1,V_2) are eigenvectors (by construction) to S, with eigenvalues k_1,k_2, also

$S_{12}=s(F_1)\cdot F_2 \neq 0$, so $\|V_1\|, \|V_2\| \neq 0$, so let $E_1:= V1 \setminus \|V_1\|$, $E_2 := V_2 \setminus\|V_2\|$ and $E_3 =F_3$ or $E_3=E_1 \times E_2$ to be principle frame field. ■

So by having E_1,E_2,E_3 , we have $S(E_1)=k_1E_1$, $S(E_2)=k_2 E_2$ so by corollary 2, §2.4, we have

$S(E_1) = \omega_{13}(E_1)E_1 + \omega_{23}(E_1)E_2 =k_1E_1$

$S(E_2) = \omega_{13}(E_2)E_1 +\omega_{23}(E_2)E_2 = k_2E_2$

So we have $\omega_{13} (E_1)= k_1$ $\omega_{23}(E_1)=0$ and also $\omega_{13}(E_2)=0$ and $\omega_{23}(E_2)=k_2$

So by the basis equation, and lemma 1 we have $\omega_{13}=k_1 \theta_1$ and $\omega_{23}=k_2 \theta_2$ (*)

Theorem 5:

If E_1,E_2,E_3 are adapted frame field on $M \subseteq R^3$, then

1) $d k_1(E_2) = (k_1-k_2) \omega_{12}(E_1)$

2) $d k_2(E_1) = (k_1-k_2) \omega_{12}(E_2)$

Proof:

By (*) above, $d (k_1 \theta_1) = d(\omega_{13}) = \omega_{12} \wedge \omega_{23} = \omega_{12} \wedge k_2 \theta_2$

So $d k_1\wedge \theta_1 +k_1 \wedge d\theta_1 = k_2 \omega_{12}\wedge \theta_2$, but $d \theta_1= \omega_{12}\wedge \theta_2$ (by structural equation)

$d k_1\wedge \theta_1 + k_1 \wedge d\theta_1 = k_2 \omega_{12}\wedge \theta_2$

$dk_1\wedge \theta_1 = (k_2-k_1) \omega_{12}\wedge \theta_2$, so apply this 2-form to (E_1,E_2)

$dk_1\wedge \theta_1(E_1,E_2)= (k_2-k_1) \omega_{12}\wedge \theta_2 (E_1,E_2)$

$dk_1 \wedge \theta_1(E_1)$-$(dk_1 \wedge \theta_1(E_2)$= $(k_2-k_1)\ \omega_{12} \wedge \theta_2 (E_1)$-$(k_2-k_1)\ \omega_{12} \wedge \theta_2 (E_1)$

$0-dk_1(E_1)=(k_2-k_1)\ \omega_{12}(E_1) -0$

So $dk_1(E_1)=(k_2-k_1)\ \omega_{12}(E_1)$ and by the same way, we can derived the second equation.

■

Chapter 3: Curvature

§ 3.1:Geometric Surfaces:

In this section, we prove results about the geometric object in abstract setting, throughout this chapter, let M denote a manifold.

Definition:

Let V be R-vector space, a map $< , >: V\times V \rightarrow R$ is called inner product if

i) $< , >$ is an R-bilinear map.

ii) $<v,w>=<w,v>$, $\forall$ v,w $\in$V (Symemetry)

iii) $<v,w>\geq 0$ and $<v,w>=0$ $\Leftrightarrow$ v=w

Note:

1- The length of a vector space v ∈ V is defined by $||v||:=\sqrt{< v, v >}$

2- Schwarz' inequality holds true $<v,w> \leq ||v||\,||w||$

3- The angle between two vectors v,w ∈ V is $\cos\theta=\frac{<v,w>}{||v||||w||}$

Definition: (Geometric Surface)

A geometric surface is a manifold M tougher with an inner product on its tangent plane T_pM

Note:

Let $g_p : T_pM \times T_pM \ni (v,w) \rightarrow <v,w>_p \in R$ be an inner product, then M is a geometric surface.

So Surface (Manifold) + Metric tensor (existence of such g) =

Geometric Surface

Construction of Tensors and Geometric objects:

1- Conformal change:

Let M be a geometric surface, let N be another subset of M (which we want to construct a metric on it), then let $h: M \rightarrow R^+$ be a map $h(p)\neq 0$, then

$<v,w>_N := <v,w>_M \backslash h^2(p)$

So we got another geometric surface.

2- Pullback:

Let F:M→N, where N is geometric surface with metric tensor g, then $F^*(g)$ defines an inner product on M as follows $<v,w>_M := <F_*(v),F_*(w)>_N$

3-Coordinate Description:

If M is a surface without geometry, let E,G,F defined as we did in the lecture

So for any patch σ on M, $E := <\sigma_s, \sigma_s>$ $F:=<\sigma_s, \sigma_t>$ and $G:= <\sigma_t, \sigma_t>$

where $<\sigma_s, \sigma_t> := av_1w_1 + b(v_1w_2+v_2w_1) + cv_2w_2$ $ac-b^2\geq 0$, $a,c>0$

Let M be a geometric surface, now the frame field is defined as before, E_1,E_2 are orthogonal i.e. $<E_1,E_1> = 1 = <E_2,E_2>$ and $<E_1,E_2>= 0$

Also the 1-dual form defined analogously $\theta_i(E_{Ej})=\delta_{ij}$ and $d\theta_1 = \omega_{12} \wedge \theta_2$ and $d\theta_2 = \omega_{21} \wedge \theta_1$ (the first structural equation)

Now Let E'_1,E'_2 be another set of frame field on M, then

$E'_1=\cos\theta\, E_1+\sin\theta\, E_2$ and so there is two candidate for E'_2

either $E'_2=-\sin\theta\, E_1+\cos\theta\, E_2$ --- we called same orientation

or $E'_1= -\sin\theta\, E_1-\cos\theta\, E_2$ --- we called opposite orientation

Now let θ_1', θ_2', ω'_{12} which are gotten from the new frame E'_1, E'_2, then the following lemma relate them θ_1, θ_2, ω_{12} of E_1,E_2

Lemma 1:

Let E_1,E_2,E_1',E_2' be a frame field, then

1) E_1',E_2' has the same orientation, then

$$\omega'_{12} = \omega_{12} + d\theta \text{ and } \theta'_1 \wedge \theta'_2 = \theta_1 \wedge \theta_2$$

2) E_1',E_2' has the opposite orientation, then

$\omega'_{12} = -(\omega_{12} + d\,\theta)$ and $\theta'_1 \wedge \theta'_2 = -\,\theta_1 \wedge \theta_2$

Proof:

1) $E'_1 = \cos\theta\, E_1 + \sin\theta\, E_2$

$E'_1 = -\sin\theta\, E_1 + \cos\theta\, E_2$ then

$\theta_1 = \cos\theta\, \theta'_1 - \sin\theta\, \theta'_2$ (i)

$\theta_2 = \sin\theta\, \theta'_1 + \cos\theta\, \theta'_2$ (ii)

$d\,\theta_1 = -\sin\theta\, d\,\theta \wedge \theta'_1 + \cos\theta\, d\,\theta'_1 - \cos\theta\, d\,\theta \wedge \theta'_2 - \sin\theta \wedge d\,\theta'_2$

by the structural equation

$d\,\theta_1 = (\omega'_{12} - d\,\theta) \wedge (\sin\theta\, \theta'_1 + \cos\theta\, \theta'_2)$

$= (\omega'_{12} - d\,\theta) \wedge \theta_2$ --- (*)

and by the same way $d\,\theta_1 = -(\omega'_{12} - d\,\theta) \wedge \theta_1$

So by the structural equation to $d\theta_1 = \omega_{12} \wedge d\,\theta_2$,then in (*)

$\omega'_{12} = \omega_{12} + d\,\theta$ and $\theta'_1 \wedge \theta'_2 = \theta_1 \wedge \theta_2$ from (i) &(ii) ■

Note:

The geometric surface is called 2-dimensional Riemann Manifold. A Manifold of any dimension is called Riemann Manifold.

§ 3.2:Gaussain Curvature:

In this section, we will give a definition of Gauss curvature regardless (independent of the shape operator) which we don't have any definition for the shape operator in the geometric surface.

Theorem 1:

Let M be any geometric surface, there exist a unique map $K: M \to R$ such that for every frame field on M, the second structural equation holds i.e.

$d\omega_{12} = -K\, \theta_1 \wedge \theta_2$, The map K is called the Gaussian curvature

Proof:

Let E_1,E_2 be a frame field, then by the basis formula in §2.4, there exist a unique function K such that $d\omega_{12} = -K\, \theta_1 \wedge \theta_2$

now Let E'_1,E'_2 be another frame field with $d\omega'_{12} = -K'\, \theta'_1 \wedge \theta'_2$, now let E_1,E_2 & E'_1,E'_2 have the same orientation, then by lemma 1, §3.1 we have

$\omega'_{12} = \omega_{12} + d\,\theta \Rightarrow d\,\omega'_{12} = d\omega_{12}$ and $\theta'_1 \wedge \theta'_2 = \theta_1 \wedge \theta_2$, so

$K'\, \theta'_1 \wedge \theta'_2 = K'\, \theta_1 \wedge \theta_2 = K\ \theta_1 \wedge \theta_2 \Rightarrow K=K'$

and for opposite orientation we wil have $d\,\omega'_{12} = -\, d\omega_{12}$ but also $\theta'_1 \wedge \theta'_2 = -\, \theta_1 \wedge \theta_2$, so again $K=K'$ ∎

Example:

Let $M := R^2$, $E_1 = (1,0)$, $E_2=(0,1)$, then $\theta_1=dx_1$, $\theta_2=dx_2$, then $d\,\theta_1=d(dx)=0$ & $d\,\theta_2=d(dy)=0$, now $\omega_{12}=0$ so $d\omega_{12}=K\,dx \wedge dy=0$ so $K=0$, as we expected to measure the flatness of R^2.

Corollary 2:

Let $M:=R^2$, let h be differentiable map defining a metric tensor $<v,w> = v\cdot w \setminus h^2(p)$ (conformal construction), then

$K=h(h_{xx}+h_{yy}) - (h^2_x+h^2_y)$

Proof:

Let N be the new object with that metric, then $id:R^2 \rightarrow N$ is a patch with $E=G=1\setminus h^2, F=0$ and so $K=h^2\, \Delta\log(h)= h(h_{xx}+h_{yy}) - (h^2_x+h^2_y)$ ∎

Now in §3.1, we haven seen that given a map $F:M\rightarrow N$, we can pull back a metric tensor on N back to M to give M a geometry, now the question can we ush forward a metric from M to N such that N gets a geometry, the following proposition gives the required consistency condition.

Proposition 3:

Let $F:M\rightarrow N$, M is a geometric surface, N is a surface without geometry, suppose $F(p_1)=F(p_2)$ implies there is an isometry $G_{12}:N(p_1)\rightarrow N(p_2)$ such that

$F^\circ G_{12}=F$ and $G_{12}(p_1)=p_2$, $N(p_1)$:= neighborhood of p_1, then there is a unique metric tensor on N that makes F a local isometry.

Proof:

Let $p_1 \in M$ such that $F(p_1)=q \in N$, to have F a local isometry, so let v,w are tangent to N in p_1, then $F_*(v_1)=v$ and $F_*(w_1)=w$, for some $v_1,w_1 \in M$ (exist to have F isometry), so we must have define $<v,w>_N := <v_1,w_1>_M$

now let $p_2 \in M$ be another point in M in which $F(p_2)=q$, then if $v_2,w_2 \in T_p(M)$ such that $F_*(v_2)=v$ and $F_*(w_2)=w$, then $<v_2,w_2>_M = <v_1,w_1>_N$ (*)

Let G_{12} as in the proposition , then $FG_{12}=F$, so $F_*G_*=F_*$, so $G_*(v_1)=v_2$ and $G_*(w_1)=w_2$, since G_{12} is an isometry, then (*) holds ■

Example:

Let PR^3 be the projective plane of R^3 i.e. by identifying antipodal points on the sphere, i.e. $PR^3 = S^2/a\sim -a$, then

$A:S^2 \rightarrow PR^3$ with $A(p)=-p$ is a map such that FA=F and A is isometry, so by the previous proposition both object are the same and so $K(PR^3)=1\backslash r^2$ =gauss curvature of the sphere.

§ 3.3:Levi Civita Connection and The Fundamental Theorem of Riemannian Geometry for dimension 2):

The idea behind this section is to define a similar notion of covariant derivative ∇ of R^3 , so for any geometric surface M together with inner product g, we want to show an existence of such notion of covariant called Levi Civita connection.

so as for R^3, the covariant derivative on any geometric surface is assigns for two vector field V,W a new vector $\nabla_V W$ which is a rate of change at a point $p \in M$ of W in the direction of V.

Required:

Covariant derivative $\nabla_V W$ satisfying the linear and Leibnizian properties (1)-(4) of corollary 1 of §1.2.

i.e. let $v,w \in T_pM$, let $F,G:M \rightarrow M$, then for $a,b \in R$, $f:M \rightarrow R$

(1) $\nabla_{av+bw}F=a\nabla_v F +b\nabla_w F$

(2) $\nabla_v(aF+bG)=a\nabla_v F +b\nabla_v G$

(3) $\nabla_v(f F)= \nabla_v f F(p) + f(p) \nabla_v F$

(4) $\nabla_v(F\ G) = \nabla_v F\ G + G\ \nabla_v F$

Let E_1, E_2 be frame field, ω_{12} be connection form, then we know that ω_{12} measure the rate at which E_1 turning toward E_2, so also we have to ask for

$\omega_{12} = < \nabla_V E_1 , E_2 >$ (*)

So if we ask for (1)-(4) o be satisfied together with (*), we can completely compute $\nabla_V W$ as the following lemma says

<u>Lemma 1</u>:

Assume that there is a covariant derivative with the properties (1)-(4) and (*) holds for a frame frame field E_1&E_2, then ∇ satisfied the <u>connection equation</u>

$\nabla_v E_1 = \omega_{12}(v)E_2$ and $\nabla_v E_2 = \omega_{21}(v)E_1$ and furthermore if $w=f_1E_1+f_2E_2$, then
$\nabla_V W=((\nabla_v f_1+f_2\omega_{21}(v))E_1+ (\nabla_v f_2+f_1\omega_{12}(v))E_2$ "Covariant derivative Formula"

<u>Proof</u>:

Now $\nabla_v E_1=c_1E_1+c_2E_2$, now $c_i = <\nabla_v E_i,E_i>$ (i=1,2)

so $\nabla_v E_1 = <\nabla_v E_1,E_1> E_1 + <\nabla_v E_1,E_2> E_2$

now $\nabla_v <E_1,E_1> = 2 <\nabla_v E_1,E_1> =0$, so

$\nabla_v E_i = <\nabla_v E_1,E_2> E_2 = \omega_{12}(v) E_2$ by (*)

and hence similarly $\nabla_v E_2 = \omega_{21}(v)E_1$

Now let $w=f_1E_1+f_2E_2$, so $\nabla_v w= \nabla_v(f_1E_1+f_2E_2)$

$\nabla_v w= \nabla_v(f_1)E_1 + f_1\nabla_v(E_1) +\nabla_v(f_2)E_2 + f_2\nabla_v(E_2)$

$= \nabla_v(f_1)E_1 + f_1\ \omega_{12}(v)E_2+\nabla_v(f_2)E_2 + f_2\ \omega_{21}(v)E_1$

$= (\nabla_v(f_1)+ f_2\ \omega_{21}(v))\ E_1 + + (\nabla_v f_2+f_1\omega_{12}(v))E_2$ ■

This is main theorem of this project which shows the existence of Levi citiva connection and is called the Fundamental theorem of Riemannian Geometry, unfortunate we proved it here only for case dim=2

<u>Theorem 2</u>: (The Fundamental theorem of Riemannian Geometry for dim 2)

Let M be a geometric surface, then there exist a unique covariant derivative ∇ with the linear and Leibnizian properties (1)-(4) and (*) for every frame field.

Proof:

I- **Uniqueness:**

By the previous lemma 1, it says if there exist such covariant with properties (1)-(4) and (*), then this can by completely determined the covariant, so there exist at most one covariant derivative ∇.

II- **Existence**:

Step 1: Local Definition

Let E_1, E_2 be a frame field on a region N of M, now we define

$\nabla_V W := (\nabla_v(f_1) + f_2\, \omega_{21}(v))\, E_1 + (\nabla_v f_2 + f_1 \omega_{12}(v)) E_2$, then clearly

(1),(2),(4) and (*) is satisfied, now we want to show property (3) (Leibnizian Property) namely to show $\nabla_V (fY) = \nabla_V f\, Y + f\, \nabla_V Y$

So let $Y := g_1 E_1 + g_2 E_2$, then $f\, Y = f\, g_1 E_1 + f\, g_2 E_2$

$\nabla_V (f\, Y) = (\nabla_v(fg_1) + f\, g_2\, \omega_{21}(v))\, E_1 + (\nabla_v(fg_2) + fg_1\, \omega_{12}(v)) E_2$

$= (\nabla_v(f)\, g_1 + f\, \nabla_V(g_1) + f\, g_2\, \omega_{21}(v))\, E_1 + (\nabla_v(f) g_2 + f\, \nabla_V (g_2) + fg_1\, \omega_{12}(v)) E_2$

$= \nabla_V f\, Y + f\, \nabla_V\, Y$

And (4),(*) will follos with $W=E_1, f_1=1, f_2=0$

Step 2: Consistency

For two different frame field, do the local definition agree & give the same covariant derivative?

So let ∇' be the covariant derivative derived from E'_1, E'_2 on domain N' on M, we have to show $\nabla_V W = \nabla'_V W$ on $N \cap N'$

So we have to show the statement on the basis, i.e.

$\nabla_V E'_1 = \nabla'_V E'_1$ and $\nabla_V E'_2 = \nabla'_V E'_2$

So w.l.o.g. assume the two frame have the same orientation, then

$E'_1 = \cos\theta\, E_1 + \sin\theta\, E_2$ and so we apply ∇_V , then

$\nabla_V E'_1 = \sin\theta\, (-\nabla_V\, \theta + \omega_{12}) E_1 + \cos\theta\, (\nabla_V\, \theta + \omega_{12})\, E_2$

By Lemma 1, § 3.1, $\omega'_{12} = \omega_{12} + d\theta = \omega_{12} + \nabla_V\theta$, substitute above yield

$\nabla_V E'_1 = \omega'_{12} (-\sin\theta\, E_1 + \cos\theta\, E_2) = \omega'_{12} E'_2 = \nabla'_V E'_1$

And by the same way $\nabla_V E'_2 = \nabla'_V E'_2$ ■

Example:

Let ∇ be the usual covariant of R^2, let e_1,e_2 be the natural basis for R^2, then $\omega_{12}=0$ and so for any vector field $w=f_1e_1+f_2e_2$, we have $\nabla_V W=(\nabla_V f_1, \nabla_V f_2)$.