Last time: the Divergence Theorem

Assume \( \mathbf{E} \) is a vector field with continuous first derivatives on an open set \( D \subset \mathbb{R}^3 \). Assume that \( B \subset D \) is a “nice” solid. Then

**Theorem (Divergence Theorem)**

\[
\iiint_B \text{div} \, \mathbf{E} \, dV = \iint_{\partial B} \mathbf{E} \cdot d\mathbf{S}.
\]

**Example.** Assume \( D = \mathbb{R}^3 \setminus \{(0, 0, 0)\} \) is everything except the origin. Suppose that \( \text{div}\mathbf{E} = 0 \) on \( D \). Let \( S_r = \{x^2 + y^2 + z^2 = r^2\} \) and let \( S'_r = \{(x - 3)^2 + (y - 1)^2 + z^2 = r^2\} \). Find \( I_1 = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} \) and \( I_2 = \iint_{S'_1} \mathbf{E} \cdot d\mathbf{S} \).

(a) Not enough information to find either.
(b) \( I_1 = I_2 = 0 \).
(c) Not enough information to find \( I_1 \); but \( I_2 = 0 \).
(d) \( I_1 = 0 \); not enough information to find \( I_2 \).
Solution

Let $B$ be any solid that doesn't contain $(0, 0, 0)$, so that $B \subset D$. (e.g. the solid inside the sphere $S'_r$.)

By the Divergence theorem,

$$\iiint_{\partial B} \mathbf{E} \cdot d\mathbf{S} = \iiint_{B} \text{div}\mathbf{E} \ dV = \iiint_{B} 0 \ dV = 0.$$ 

Note that this argument doesn't work for $S_1$, because the solid inside of $S_1$ contains $(0, 0, 0)$, so $\mathbf{E}$ is not defined over the whole solid, and the Divergence Theorem doesn't apply.

In fact, we will see later how to calculate $\iiint_{S_1} \mathbf{E} \cdot d\mathbf{S}$ explicitly for a certain example of $\mathbf{E}$, and we will see that it's not 0.
Announcements

- Final exam is next Friday. Register for conflict by Monday.
- Office hours/review session next week:
  - Ordinary office hours Tuesday 11–11:50am.
  - Extra office hours Wednesday evening (probably 6–7pm, maybe 7–8pm—it’s fine with me if you bring your dinner). AH 341
  - Extra office hours Thursday 12–1pm. AH 341
  - Possibly office hours also on Friday, but I can’t confirm yet.
  - Come with questions (or you can listen to other people’s questions).
- Fact: today is our ante-penultimate lecture. Wednesday was our pre-ante-penultimate lecture, but I forgot to say so.
Electric field and electric flux

Given a particle of charge \( Q \) at \((0, 0, 0)\), its electric field is

\[
E(x, y, z) = \frac{Q}{4\pi \varepsilon_0 (x^2 + y^2 + z^2)^{\frac{3}{2}}} \langle x, y, z \rangle
\]

or equivalently

\[
E(r) = \frac{Q}{4\pi \varepsilon_0 |r|^3} r.
\]

Inverse square law

This means that the force experienced by a particle of charge \( q \) at position \( r \) is \( qE(r) \).

Where is the vector field \( E \) defined?

(a) all of \( \mathbb{R}^3 \)

(b) everywhere except \((0, 0, 0)\)

(c) everywhere except the z-axis, \( \{x = y = 0\} \)

(d) I don’t know
Practice with Gauss’ Law

Suppose we have particles of charge $Q_i$ at points $P_i$, with $Q_i = i$ for $i = 1, 2, 3, 4, 5$. Suppose that $B$ is a solid region containing $P_1$, $P_3$, and $P_4$, but not $P_2$ or $P_5$. What is

$$\int \int \int_{\partial B} \mathbf{E} \cdot d\mathbf{S}?$$

(a) 0
(b) $\frac{1}{\epsilon_0}$
(c) $\frac{2}{\epsilon_0}$
(d) $\frac{4}{\epsilon_0}$
(e) $\frac{8}{\epsilon_0}$
The enclosed charge is $Q_1 + Q_3 + Q_4 = 1 + 3 + 4 = 8$.

So by Gauss’ Law,

$$\int \int_{\partial B} \mathbf{E} \cdot d\mathbf{S} = \frac{8}{\varepsilon_0}$$