Let \( \mathbf{F} = \langle P, Q, R \rangle \) be a vector field on \( D \subset \mathbb{R}^3 \).

\[
\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle;
\]
\[
\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z.
\]

Let \( \mathbf{F}(x, y, z) = \langle P(x, y), Q(x, y), 0 \rangle \). Compute \( \text{curl}\mathbf{F} \) and find the function \( \text{curl}\mathbf{F} \cdot \mathbf{k} \), where \( \mathbf{k} \) is the vector \( \langle 0, 0, 1 \rangle \).

(a) \( P_x - Q_y \)
(b) \( Q_x - P_y \)
(c) \( P_x + Q_y \)
(d) \( P_y + Q_x \)
(e) I don’t know how.
Physical interpretation of curl

Let $\mathbf{F}$ be a vector field on $D \subset \mathbb{R}^3$, representing the velocity of a fluid flowing through the region $D$.

For a point $P \in D$, we consider the vector $\text{curl}(\mathbf{F})(P)$.

- The line through $P$ in the direction of $\text{curl}(\mathbf{F})(P)$ is the axis of rotation of a tiny ball at point $P$.
- The direction of $\text{curl}(\mathbf{F})(P)$ is related to the direction of rotation by the right-hand rule.
- The magnitude $|\text{curl}(\mathbf{F})(P)|$ is proportional to the speed of rotation.

In particular, when $\text{curl}(\mathbf{F}) = 0$, the little ball doesn’t rotate at all; we say that $\mathbf{F}$ is irrotational at $P$.

Note: the ball can still be moving! It’s floating along the current, it’s just not spinning as it moves past the point $P$. 
Practice with curl

Let \( \mathbf{F}(x, y, z) = \langle y, 0, 0 \rangle \). By imagining a tiny ball placed at different locations in the vector field, decide whether \( \text{curl}(\mathbf{F}) \) points up, points down, or is zero.

(a) It always points up.

(b) It always points down.

(c) It’s always zero.

(d) It depends what point we look at.

(e) I don’t know.

If you’re done, calculate \( \text{curl}(\mathbf{F}) \) from the definition and see if it matches your prediction.
Physical interpretation of div

- If $\text{div} \mathbf{F}$ is positive, fluid flows out of $B$, a small ball around the point.
- If $\text{div} \mathbf{F}$ is negative, fluid flows in to $B$.
- If $\text{div} \mathbf{F}$ is zero, there is no net change: the volume of fluid coming in is equal to the volume of fluid going out. In that case, we say that $\mathbf{F}$ is incompressible.
Practice with div

Let \( \mathbf{F}(x, y, z) = \langle x, 0, 0 \rangle \). Imagine a small region around a point. Is fluid leaving the region more quickly than it is entering it? Use your observation to decide whether \( \text{div}\mathbf{F} \) is

(a) always positive.
(b) always negative.
(c) always zero.
(d) It depends on the point.
(e) I don’t know.

If you’re done, calculate \( \text{div}\mathbf{F} \) from the definition, and see if your prediction is correct.