Review: integrating vector fields over curves.
[See slides for example & solution]

Today: Green's theorem.

Recall: A path is a piecewise smooth curve.

**Fundamental theorem of calculus:**
\[
\int_a^b f'(x) \, dx = f(b) - f(a).
\]

**Fundamental theorem of line integrals:**
\[
\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).
\]
\[
\int_C f_x \, dx + f_y \, dy
\]

Derivative on the left
Boundary on the right

Today: integrate a "derivative" over a 2D region \( B \)
over the boundary curve \( \partial B \).

Assumption: \( \mathbf{F} = \langle P, Q \rangle \) has continuous first order partial derivatives on an open set \( \mathbb{D} \subset \mathbb{R}^2 \).

"\( \mathbb{D} \) is "nice": we can integrate over \( \mathbb{D} \)
- \( \partial \mathbb{D} \) is one or more simple closed paths
- orient \( \partial \mathbb{D} \) so that \( \mathbb{D} \) is always on the left.

**Theorem: [Green's Theorem]**
\[
\int_B (Qx - P_y) \, dA = \int_{\partial B} P \, dx + Q \, dy.
\]

\[\text{derivative} \quad \text{boundary} \rightarrow \int_{\partial B} \mathbf{F} \cdot d\mathbf{r} \]
Example: Find \[ \int_C xy \, dx + \frac{x^2}{2} \, dy, \] where \( C \) is the rectangle with vertices \((0,0), (3,0), (3,1), (0,1)\).

\[ \text{B = } [0,3] \times [0,1], \text{ Nice!} \]

By Green's theorem:
\[
\int_C xy \, dx + \frac{x^2}{2} \, dy = \iint_B \frac{\partial}{\partial x} \left( \frac{x^2}{2} \right) - \frac{\partial}{\partial y} (xy) \, dA
\]
\[
= \int_0^3 \int_0^1 2x - x \, dy \, dx = \int_0^3 \int_0^1 x \, dy \, dx
\]
\[
= \frac{1}{2} \int_0^3 x^2 \, dx = \frac{9}{2}.
\]

Theorem: Area of \( B = \int_{\partial B} x \, dy = -\int_{\partial B} y \, dx = \frac{1}{2} \int_{\partial B} x \, dy - y \, dx \)

Proof of (C): By Green's theorem,
\[
\frac{1}{2} \left( \int_{\partial B} x \, dy - y \, dx \right) = \frac{1}{2} \left( \iint_B \frac{\partial}{\partial x} (x) - \frac{\partial}{\partial y} (-y) \, dA \right)
\]
\[
= \frac{1}{2} \iint_B 2 \, dA = \iint_B dA = \text{Area of } B. \tag{D}
\]

(A) and (B) are similar.

1. Use (C) to find the area of the disk \( B_r = \{ x^2 + y^2 \leq r^2 \} \).
2. Let \( \mathbf{F} = \langle P, Q \rangle = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle \)

Recall that \( P_y = Q_x \). Which argument is correct?

(A) on \( C_r \), \( \langle P, Q \rangle = \langle -y/r^2, x/r^2 \rangle \)
\[
= \int_{C_r} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{r^2} \int_{C_r} x \, dy - y \, dx = \frac{2\pi r^2}{r^2} = 2\pi.
\]

(B) on \( C_r \), \( \langle P, Q \rangle = \langle -y/r^2, x/r^2 \rangle \)
\[
= \int_{C_r} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{r^2} \int_{C_r} x \, dy - y \, dx = \frac{2\pi r^2}{r^2} = 2\pi.
\]
(B) By Green's theorem

\[ \oint_{C_1} \nabla \cdot \mathbf{F} \cdot d\mathbf{r} = \iint_{B_1} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dA = 0 \]

Another example: With \( \mathbf{F} = \left< \frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right> \) as before.

Let \( C' \) be any simple closed curve in \( \mathbb{R}^2 \) enclosing \((0,0)\).

What is \( \oint_{C'} \mathbf{F} \cdot d\mathbf{r} \)?

We can't directly use Green's theorem, because \( \mathbf{F} \) isn't defined at \((0,0)\).

Indeed, choose \( r > 0 \) a small enough and consider \(-C_r\), so that \( C' \cup (-C_r) \) forms the boundary of a region \( B \).

Now use Green's theorem to calculate \( \oint_{C'} \mathbf{F} \cdot d\mathbf{r} \).

[See slides].

**Recall Theorem:** If \( D \) is simply connected and \( \mathbf{F} \) satisfies \( P_y = Q_x \), then \( \mathbf{F} \) is conservative.

We can prove this, using Green's theorem.

**Recall** \( \mathbf{F} \) is conservative \( \iff \int_C \mathbf{F} \cdot d\mathbf{r} \) is path independent

\( \iff \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \) for any closed curve \( C \).

Note that any closed curve \( C \) can be broken into simple closed curves

\[ \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \sum_{C_i} \oint_{C_i} \mathbf{F} \cdot d\mathbf{r} \]

So it's enough to show \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \) for \( C \) a simple closed curve in \( D \).

Since \( \partial \mathbf{D} \) is simply connected, \( C = \partial B \)

for BCD.
So by Green's Theorem
\[ \int F \cdot d\mathbf{r} = \iint_B (Q_x - P_y) \, dA = \iiint_B 0 \, dA = 0 \quad \Box \]

**Why is Green's theorem true?**

\[ \int_C P \, dx + Q \, dy = \iint_B (Q_x - P_y) \, dA \]

Let's show that (*) is true for a region of type I.

\[ B = \{ (x, y) \mid a \leq x \leq b, \quad g_1(x) \leq y \leq g_2(x) \} \]

\[ = \int_C P \, dx = \int_B \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} \, dy \, dx \]

\[ = \int_a^b P(x, g_1(x)) - P(x, g_2(x)) \, dx \]

(by F.T.C.)

On the other hand

\[ \int_C P \, dx = \int_{C_1} P \, dx + \int_{C_2} P \, dx + \int_{C_3} P \, dx + \int_{C_4} P \, dx \]

\[ = \int_a^b P(t, g_1(t)) \, dt \]

(using parametrization of \( C_1 \) given by \( \mathbf{r}(t) = \langle t, g_1(t) \rangle \))

\[ = \int_a^b P(t, g_1(t)) \, dt \]

So the two sides are equal.

Likewise we prove that \( \int_C Q \, dy = \iint_B \frac{\partial Q}{\partial x} \, dA \) for \( B \) of type II.

So we need to divide our region \( B \) into small regions that are hom. of type I and type II.