MATH 241 - LECTURE 25 - TRIPLE INTEGRALS (§15.7)  

Monday, 25 March 2019

Last time - Polar coordinates.

1) Calculate the area of the leaf (of the four-leafed rose)

\[ D = \{ (r, \theta) \mid -\pi/4 \leq \theta \leq \pi/4, \quad 0 \leq r \leq \cos 2\theta \} \]

Today: TRIPLE INTEGRALS (§15.7).

Assumptions: all functions are continuous (where defined)

- Given \( E \subset \mathbb{R}^3 \) bounded and \( f: E \to \mathbb{R} \) continuous, we define the integral of \( f \) over \( E \) similarly to how we defined double integrals:

\[
\iiint_E f \, dV = \iiint_D g \, dA.
\]

Geometric interpretations

1) \( \iiint_E f \, dV = \) (average value of \( f \) on \( E \)) \( \cdot \) (volume of \( E \))

\( \Rightarrow \) 1b) when \( f = 1 \), average of \( f \) is 1

\( \Rightarrow \) \( \iiint_E dV = \) volume of \( E \).

2) Given a solid occupying the space defined by \( E \), with density \( g(x, y, z) \), \( g(x, y, z) > 0 \) at \( (x, y, z) \in E \):

Total mass: \( m = \iiint_E g(x, y, z) \, dV \)

Centre of mass: \( (\bar{x}, \bar{y}, \bar{z}) \), where \( \bar{z} = \frac{\iiint_E z \, g(x, y, z) \, dV}{m} \) etc.
Moment of inertia about x-axis:

\[ I_x = \iiint_E (y^2 + z^2) \, f(x,y,z) \, dV \]

\[ = \text{(distance from point} \ (x,y,z) \ \text{to} \ x\text{-axis})^2 \]

etc.

**How do we calculate triple integrals?**

* we have versions of Fubini's theorem for certain regions,

(similar to regions of Type I and Type II in \( \mathbb{R}^2 \)).

Suppose \( E \) can be written as

\[ E = \{ (x,y,z) \mid (x,y) \in D, \quad \min(u_1(x,y)) \leq z \leq \max(u_2(x,y)) \} \]

Then

\[ \iiint_E f \, dV = \iiint_D \left( \int_{u_1(x,y)}^{u_2(x,y)} f \, dz \right) \, dA \]  \hspace{1cm} (\ast) \]

- To find \( D \) - look for the shadow of \( E \) if a light shines down on it.

- To find \( u_1, u_2 \): given \( u(x,y) \in D \), \( E \cap \{ z = x_0, y = y_0 \} \)

\( u(x,y) \) is a vertical line segment with endpoints \( u_1(x_0,y_0) \) and \( u_2(x_0,y_0) \).

* if \( D \) is also flat nice: we can calculate \((\ast)\):

* e.g. \( D = \{ (x,y) \mid a \leq x \leq b \}

\[ g_1(x) \leq y \leq g_2(x) \]  \hspace{1cm} \text{Type I} \]

then \( E = \{ (x,y,z) \mid a \leq x \leq b \}

\[ g_1(x) \leq y \leq g_2(x) \] \hspace{1cm} \text{and:}

\[ u_1(x,y) \leq z \leq u_2(x,y) \]

\[ \iiint_E f \, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z) \, dz \, dy \, dx \]
Example: Let \( E \) be the region bounded by the paraboloid \( y = x^2 + z^2 \) and the plane \( z = 4 \).

- **Shadow on xy-plane:**
  \[
  D_1 = \left\{(x, y) \mid -2 \leq x \leq 2, \quad 2 \leq y \leq 4\right\}
  \]

  \[\Uparrow \] \( E = \left\{(x, y, z) \mid (x, y) \in D_1 \right\} \)

- **Shadow on \((x, z)\) plane:**
  \[
  D_2 = \left\{(x, z) \mid x^2 + z^2 \leq 4\right\}
  \]

  \[\Uparrow \] \( E = \left\{(x, y, z) \mid (x, z) \in D_2, \quad x^2 + z^2 \leq y \leq 4\right\} \).

Now let's use the second description to calculate the volume of \( E \).
(See slides for solution.)

Another example. Let \( E \) be the region bounded by the four planes

- \( x = 2 \)
- \( z = 0 \)
- \( y = 2 \)
- \( x + y = 2z = 2 \).

- **Face on the xy plane \((z = 0)\):**

- **Remaining vertex:** \( x = 2, y = 2, \quad x + y = 2z = 2 \)
  \( \Rightarrow \) \( z = 1 \)

\[\square \text{Write} \quad D = \left\{(x, y) \mid \, ? \leq x \leq ? \right\} \]
\( \quad \, ? \leq y \leq ? \)

\[\square \text{Write} \quad E = \left\{(x, y, z) \mid (x, y) \in D \right\} \]
\( u_1(x, y) \leq z \leq u_2(x, y) \).
Given \((x, y, z)\), write
\[
    x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.
\]

**Example:** Let \(E\) be the region bounded by
\[
    x^2 + y^2 = 1
\]
\[
    z = 1 - x^2 - y^2
\]
\[
    z = z
\]
\[
    x^2 + y^2 = 1 \quad \text{i.e.} \quad r^2 = 1 \quad \text{i.e.} \quad r = 1
\]
\[
    z = 2
\]
\[
    z = 1 - x^2 - y^2 \quad \text{i.e.} \quad z = 1 - r^2
\]

So \(E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 1 - r^2 \leq z \leq z\}\).