Last time: iterated integrals

Let \( D = [0, 2] \times [-3, 1] \). Find \( \iint (3x^2 + 3y^2) \, dA \).

(a) -12
(b) 42
(c) 88
(d) Some other number
(e) I don’t know

(If you’re done, try integrating using the opposite order of integration to what you used the first time. You should get the same answer.)
\[
\int \int_D (3x^2 + 3y^2)\,dA = \int_0^2 \int_{-3}^1 (3x^2 + 3y^2)\,dy\,dx
\]
\[
= \int_0^2 \left[ 3x^2y + y^3 \right]_{-3}^1 \,dx
\]
\[
= \int_0^2 \left[ (3x^2 + 1) - (-9x^2 - 27) \right] \,dx
\]
\[
= \int_0^2 12x^2 + 28 \,dx
\]
\[
= \left[ 4x^3 + 28x \right]_0^2
\]
\[
= 32 + 56 = 88.
\]
Solution (opposite order)

\[
\int \int_D (3x^2 + 3y^2) \, dA = \int_{-3}^{1} \int_{0}^{2} (3x^2 + 3y^2) \, dx \, dy
\]
\[
= \int_{-3}^{1} \left[ x^3 + 3xy^2 \right]_0^2 \, dy
\]
\[
= \int_{-3}^{1} \left[ 8 + 6y^2 \right] \, dy
\]
\[
= \left[ 8y + 2y^3 \right]_{-3}^{1}
\]
\[
= (8 + 2) - (-24 - 54)
\]
\[
= 88.
\]
Recall: Fubini’s Theorem

**Theorem**

Let \( f \) be a continuous function on \( D = [a, b] \times [c, d] \). Then

\[
\int \int_D f(x, y)dA = \int_c^d \int_a^b f(x, y)dx \, dy = \int_a^b \int_c^d f(x, y)dy \, dx.
\]

More generally, this is true if \( f \) is bounded and is continuous except at a finite number of smooth curves, provided that the iterated integrals exist.
Regions of Type I

We say \( D \subset \mathbb{R}^2 \) is of Type I if it is of the form

\[
D = \{(x, y) \mid a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(y)\},
\]

where \( g_1, g_2 : [a, b] \to \mathbb{R} \) are continuous functions.

**Theorem**

Let \( f(x, y) \) be a continuous function on a region \( D \) of type I as above. Then

\[
\iint_D f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.
\]
Practice with regions of type II

Recall the region $D$ enclosed by the lines $x = 0$, $y = 1$, and the curve $y = x^2$.

To show that $D$ is a region of type II, we need to find numbers $c$ and $d$ and continuous functions $h_1$, $h_2$ on the interval $[c, d]$ such that

$$D = \{(x, y) \mid a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(y)\}.$$  

(a) I don’t know what to do.
(b) I’m working on it.
(c) I have answers, but they don’t match with my neighbour’s.
(d) We agree.
Solution

To find the interval $c$ and $d$ we look at the “shadow” produced by the region $D$ on the $y$-axis. (Pretend a big light is shining from the far right side.)

So we see that $[c, d] = [0, 1]$.

Now given a point $y_0$ in this interval, what values can $x$ take?

$x$ must be larger than $\sqrt{y_0}$ and smaller than 1.

So $h_1(y) = \sqrt{y}$ and $h_2(y) = 1$. 
Integrating over a region of type II

Let \( D = \{(x, y) \mid 0 \leq y \leq 1 \text{ and } \sqrt{y} \leq x \leq 1\} \). How would you find the area of \( D \)? Fill in the blanks in the following formula:

\[
\text{Area of } D = \int_{?}^{?} \int_{?}^{?} \, d? \, d?.
\]

(a) I don’t know what to do.
(b) I’m working on it.
(c) I have answers, but they don’t match with my neighbour’s.
(d) We agree.
\[ D = \{(x, y) \mid 0 \leq y \leq 1 \text{ and } \sqrt{y} \leq x \leq 1\} \]

The area of \( D \) is \( \iint_D 1dA \); since \( D \) is of type II we have

\[
\iint_D 1dA = \int_0^1 \int_{\sqrt{y}}^1 1dx\,dy.
\]
Practice with integrating over polar rectangles

Let \( D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4 \text{ and } 0 \leq y\} \) as before. What is \( \int\int_D y \, dA \)?

(a) 0  
(b) \( \frac{14}{3} \)  
(c) 3  
(d) \( 3\pi y^2 \)  
(e) I don’t know.
Solution

Since $D$ is a polar rectangle of the form with $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi$, we have

\[
\int \int_{D} ydA = \int_{0}^{\pi} \int_{1}^{2} (r \sin \theta) r dr d\theta 
\]

\[
= \int_{0}^{\pi} \int_{1}^{2} r^2 \sin \theta dr d\theta 
\]

\[
= \int_{0}^{\pi} \left[ \frac{1}{3} r^3 \sin \theta \right]_{1}^{2} d\theta 
\]

\[
= \int_{0}^{\pi} \frac{7}{3} \sin \theta d\theta 
\]

\[
= \left[ -\frac{7}{3} \cos \theta \right]_{0}^{\pi} 
\]

\[
= \frac{14}{3}. 
\]