Let $C_1 = \{(x, y) \mid x^2 + y^2 = 1 \text{ and } y \geq 0\}$.
Let $C_2 = \{(x, y) \mid x^2 + y^2 = 1 \text{ and } y \leq 0\}$
Orient both from $(−1, 0)$ to $(1, 0)$.
Let $\mathbf{F}(x, y) = \langle y, −x \rangle$. (Note: I had the opposite sign on the slide in class, but that was wrong.)

Use $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$ to calculate

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} − \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

(a) 0
(b) $2\pi$
(c) $−2\pi$
(d) $−\pi$
(e) I don’t know what to do.

Correct answer: (b)
Solution

For $C_1$:
$F = T$, so

$$\int_{C_1} F \cdot T \, ds = \int_{C_1} |T|^2 \, ds = \int_{C_1} 1 \, ds = \pi.$$

For $C_2$:
$F = -T$, so

$$\int_{C_2} F \cdot T \, ds = \int_{C_2} (-1) \, ds = -\pi.$$

So $\int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr = \pi - (-\pi) = 2\pi$. 

So $\int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr = \pi - (-\pi) = 2\pi$. 
Example 1

Let $\mathbf{F} = \nabla f$ be a conservative vector field on $\mathbb{R}^2$ or $\mathbb{R}^3$, and let $C$ be a curve with initial point $P$ and terminal point $Q$. Assume that $\nabla f$ is continuous.

The Fundamental Theorem of Line Integrals tells us that

$$\int_C \nabla f \cdot d\mathbf{r} = f(Q) - f(P).$$

This implies that $\mathbf{F}$ is independent of path.
Let $\mathbf{F}(x, y) = \langle -y, x \rangle$.

At the beginning of class, we found two curves $C_1$ and $C_2$ with the same initial point $(-1, 0)$ and the same terminal point $(1, 0)$, but we showed that the integrals of $\mathbf{F}$ over $C_1$ and $C_2$ were not equal.

So $\mathbf{F}$ is not path independent.

**Remark:** Combining this observation with the previous slide, we can conclude that $\mathbf{F}$ is not conservative.
Is the vector field conservative?

We’re going to look at the vector field describing wind velocity. Discuss with your neighbour: is this vector field conservative? https://earth.nullschool.net/
(Remember the options below:)

(a) Yes, we think it is.
(b) No, we think it’s not.
(c) We don’t agree/we don’t know.

Answer: the vector field is not conservative. You can find circles around which the integral is not zero.
Comments on the proof

**Theorem:** For $D$ open and connected, the integral of $\mathbf{F}$ is path independent $\iff \mathbf{F}$ is conservative.

We have to prove two things.

- The integral of $\mathbf{F}$ is path independent $\Rightarrow \mathbf{F}$ is conservative.
- The vector field $\mathbf{F}$ is conservative $\Rightarrow$ the integral is path independent.

We already showed the second line, using the Fundamental Theorem of Line Integrals.
The integral of \( \mathbf{F} \) is path independent \( \Rightarrow \) \( \mathbf{F} \) is conservative.

We’re mostly going to skip the proof, but here is the main idea.

Choose any point \( P \) in \( D \).

Define \( f : D \to \mathbb{R} \) as follows.

Given any point \( Q \) in \( D \), choose a path \( C \) from \( P \) to \( Q \). We can do this because \( D \) is connected!

Now let \( f(Q) = \int_C \mathbf{F} \cdot dr \in \mathbb{R} \).

It doesn’t matter what path \( C \) we chose, because the integral is path independent!

We claim that \( \nabla f = F \), which shows that \( F \) is conservative.