Integrating functions over curves

Recall that for a (smooth) curve $C$ parametrized by a vector-valued function $\mathbf{r}$ over an interval $[a, b]$, and for a function $f : C \to \mathbb{R}$, we have

$$\int_C f \, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)| \, dt.$$  

This formula works whether $C$ is a plane curve ($\mathbf{r} : [a, b] \to \mathbb{R}^2$) or a space curve ($\mathbf{r} : [a, b] \to \mathbb{R}^3$).

Compute $\int_C x^2 z \, ds$ where $C$ is the line segment from $(0, 6, -1)$ to $(4, 1, 5)$.

(a) $\frac{56}{3} \sqrt{77}$

(b) $\frac{14}{3} \sqrt{77}$

(c) $\frac{56}{3} \sqrt{15}$

(d) $\frac{14}{3} \sqrt{15}$

Correct answer: (a)
Announcements

- **Midterm 2** is on **Tuesday, March 12** at 7pm.
  - Deadline to request a spot in the conflict exam is **next Tuesday, March 5**.

- **Register your i-clicker!** Deadline is **this Saturday, March 2, at 5pm**.
  - Check on Moodle: if you do not see any i-clicker grades, your registration has not gone through. Email me with your name, i-clicker number, and netid.

- **Thanks for your feedback.**
  - Changes: bigger chalk, more examples, more slides when possible.
  - Please continue to provide feedback (by email or anonymously e.g. through your TA).
An example of a vector field

https://earth.nullschool.net/
Matching a vector field with its plot

(a) \( \mathbf{F}(x, y) = \langle \sin(x), 1 \rangle \)
(b) \( \mathbf{F}(x, y) = \langle 1, \sin(y) \rangle \)
(c) \( \mathbf{F}(x, y) = \langle 1, \cos(y) \rangle \)
(d) \( \mathbf{F}(x, y) = \langle \sin(y), 1 \rangle \)
(e) I don’t know how

Correct answer: (b)
Let \( r(t) = \langle t, t^2 \rangle, \ t \in [0, 1]\), and let \( F(x, y) = \langle y, x \rangle \). Sketch the curve and vector field. What can you say about \( \int_C F \cdot dr \)?

(a) It’s positive.
(b) It’s negative.
(c) It’s zero.
(d) It’s not defined.
(e) I don’t know how to say anything about it.

Correct answer: (a)
Practice with integrating vector fields

Let \( \mathbf{r}(t) = \langle t, t^2 \rangle, \ t \in [0, 1] \), and let \( \mathbf{F}(x, y) = \langle y, x \rangle \) (as on the previous slide).

- \( \mathbf{F}(\mathbf{r}(t)) = \langle t^2, t \rangle \).
- \( \mathbf{r}'(t) = \langle 1, 2t \rangle \).

It follows that

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t^2, t \rangle \cdot \langle 1, 2t \rangle \, dt
= \int_0^1 3t^2 \, dt
= [t^3]_0^1 = 1.
\]

(Note that this is positive.)
Practice with integrating vector fields

Let $C$ be parametrized by $\mathbf{r}(t) = \langle t, 2t \rangle$, $t \in [0, 1]$. Let $\mathbf{F}(x, y) = \langle 1, 2y \rangle$.

What is $\int_C \mathbf{F} \cdot d\mathbf{r}$?

(a) 9  
(b) 5  
(c) 0  
(d) 20  
(e) I don’t know what to do.

(If you’re done, sketch the curve and the vector field, and check whether your answer is a reasonable one.)

Correct answer: (b)
Solution:

Let \( C \) be parametrized by \( \mathbf{r}(t) = \langle t, 2t \rangle, \ t \in [0, 1] \).

Let \( \mathbf{F}(x, y) = \langle 1, 2y \rangle \).

- \( \mathbf{F}(\mathbf{r}(t)) = \langle 1, 4t \rangle \).
- \( \mathbf{r}'(t) = \langle 1, 2 \rangle \).

\[ \Rightarrow \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 1, 4t \rangle \cdot \langle 1, 2 \rangle \, dt \]
\[ = \int_0^1 1 + 8t \, dt \]
\[ = [t + 4t^2]^1_0 \]
\[ = 5. \]