Last time: arc length \( L = \int_a^b |\ddot{r}(t)| \, dt \).

\[ \text{Example of a plane curve:} \quad \text{cycloid} \]

- Wheel of radius 1 rolls at 1 radian/second.
- \( \ddot{r}(t) \) = position of a LED on the rim at time \( t \).
- \( \text{at } t = 0, \text{ centre of the wheel is at } (0,0) \) and LED is at \((0,1)\).

\[ \text{At time } t, \text{ the centre is at } (\cos t) (t, 1). \]

\[ \text{and the vector from the centre to the LED is } <-\sin t, -\cos t>. \]

\[ = \ddot{r}(t) = \langle t, 1 \rangle + <-\sin t, -\cos t> = \langle t - \sin t, 1 - \cos t \rangle \]

Upside-down, the cycloid has special properties:

- It is an isochronous (see slides - click for videos)
- It is a brachistochrone

- This is an infinite-dimensional min/max problem
  (calculus of variations)

\[ \text{§ 16.2. INTEGRATING FUNCTIONS ALONG CURVES} \]

For today, assume curves are smooth
- i.e. \( \dot{r}(t) \) is continuous and non-vanishing.

Integration with one variable. [see slides]

- Fix \( g: [a,b] \to \mathbb{R} \)
- divide \([a,b]\) into \( n \) subintervals \([x_{i-1}, x_i]\) of size \( \Delta x = \frac{b-a}{n} \)
- for each \( i \) choose \( x_i^* \in [x_{i-1}, x_i] \)

\[ \therefore \int_a^b g(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{\infty} g(x_i^*) \Delta x \]

(if limit exists and doesn’t depend on choices \( x_i^* \))
Geometric meanings (see slides)

- \( \int_a^b g(x) \, dx = (b-a) \cdot \text{(average value of } g \text{ on } [a,b]) \)
- \( g > 0 \Rightarrow \text{area under the curve.} \)
- \( g \) is the mass of a wire; can find its center of mass.

Integration over curves:

Fix a curve \( C \subset \mathbb{R}^2 \) parametrized by \( \vec{r}(t) = \langle x(t), y(t) \rangle \), \( t \in [a,b] \).

Fix a function \( f : C \to \mathbb{R} \).

- Divide \( [a,b] \) into \( n \) subintervals \([t_{i-1}, t_i]\) of length \( \frac{b-a}{n} \)
  and choose \( t_i^* \in [t_{i-1}, t_i] \) as before.
- Let \( \Delta s_i \) be the length of \( C \) from \( \vec{r}(t_{i-1}) \) to \( \vec{r}(t_i) \).

Definition \( \int_C f ds = \lim_{n \to \infty} \sum_{i=1}^n f(\vec{r}(t_i^*)) \Delta s_i \)

(assuming the limit exists and is independent of \( t_i^* \))

Theorem: \( \int_C f ds = \int_a^b f(\vec{r}(t)) \left| \vec{r}'(t) \right| \, dt \)

\[ = \int_a^b f(\langle x(t), y(t), z(t) \rangle) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt. \]

WARNING - don't trace over the path more than once!

Proof: \( \Delta s_i \approx \left| \vec{r}(t_i) - \vec{r}(t_{i-1}) \right| \approx \left| \vec{r}'(t_i^*) \right| \Delta t \)

\[ = \lim_{n \to \infty} \sum_{i=0}^n f(\vec{r}(t_i^*)) \left| \vec{r}'(t_i^*) \right| \Delta t \]

\[ = \int_a^b f(\vec{r}(t)) \left| \vec{r}'(t) \right| \, dt. \]

Note: \( \int_C f ds \) doesn't depend on the choice of parametrization \( \vec{r} \).
(e.g. semi-circle from last time)
Geometric meanings:

1) \( \int_C f ds = (\text{average value of } f \text{ on } C) \cdot (\text{length of } C) \).

   In particular, if \( f = 1 \) everywhere
   \[ \int_C ds = \text{length of } C \]
   \[ = \int_0^1 f'(t) dt \quad \text{as we saw Friday} \]

2) If \( f > 0 \), \( f \) describes a fence of varying height over the curve \( C \)

   \[ \int_C f ds = \text{surface area of (one side of) the fence} \]

3) If \( f \) is the linear density of a wire shaped like \( C \),

   \[ \int_C f ds = \text{total mass of wire} \]

   Centre of mass of wire is at \( (\bar{x}, \bar{y}) \) where

   \[ \bar{x} = \frac{\int_C x f ds}{\int_C f ds}, \quad \bar{y} = \frac{\int_C y f ds}{\int_C f ds} \]

Example: Wire of constant density \( \rho \) over the semi-circle

\[ x^2 + y^2 = 1, \quad y > 0. \]

Use geometric reasoning to guess the most likely answer for the centre of mass.

Now let's calculate the centre of mass.

\[ f(t) = \rho \text{ constant}. \]

\[ \int_0^\pi f ds = \int_0^\pi \rho \, dt = \rho \pi. \]

\[ \int_C x f ds = \int_0^\pi \rho \cos t \, dt = [\rho \sin t]_0^\pi = 0 \quad \Rightarrow \bar{x} = 0 \]

\[ \int_C y f ds = \int_0^\pi \rho \sin t \, dt = [-\rho \cos t]_0^\pi = 2\rho \quad \Rightarrow \bar{y} = \frac{2\rho}{\pi}. \]