Is there a point on the graph \( z = \sqrt{x^2 + y^2} \) that’s closest to the point \( P = (4, 2, 0) \)? Furthest?

(a) There is a closest point and a furthest point.
(b) There is a closest point but no furthest point.
(c) There is a furthest point but no closest point.
(d) There is neither a furthest point nor a closest point.
(e) I don’t know.

Correct answer: (b)
Recall: the Extreme Value Theorem

Let \( f : D \rightarrow \mathbb{R} \) be a continuous function on \( D \), which is closed and bounded. Then \( f \) attains a maximum value on \( D \) at some point \( P \in D \), and either

- \( P \) is on the boundary of \( D \), OR
- \( P \) is a critical point of \( f \).

Today: How can we find the maximum value of \( f \) along the boundary of \( D \), without checking every single point?
Practice with Lagrange multipliers

We have the following three equations:

\[ 2x = \lambda 2x \]
\[ 2y = -\lambda 2y \]
\[ x^2 + y^2 = 4. \]

How many solutions \((x, y, \lambda)\) are there?

(a) No solutions.
(b) 2.
(c) 4.
(d) Infinitely many.
(e) I don’t know.

Correct answer: (c)
Lagrange multipliers (in three variables)

Assume \( f, g \) are functions of three variables with continuous first order partial derivatives.

If \( f(x_0, y_0, z_0) \) is the maximum value of \( f \) over the level surface \( g(x, y, z) = k \), then either

- \( \nabla g(x_0, y_0, z_0) = \langle 0, 0, 0 \rangle \) OR
- \( \nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \), for some \( \lambda \in \mathbb{R} \).

The same theorems hold for minimum values too.
Suppose \( f, g \) are functions on \( \mathbb{R}^3 \) with continuous first order partial derivatives.

Suppose that \( f \) achieves its maximum over the set \( \{ g(x, y, z) = k \} \) at the point \( P \). Which of the following is not possible?

(a) \( \nabla f(P) = \langle 0, 0, 0 \rangle, \nabla g(P) = \langle 0, 0, 0 \rangle. \)
(b) \( \nabla f(P) = \langle 0, 0, 0 \rangle, \nabla g(P) = \langle 1, 3, -2 \rangle. \)
(c) \( \nabla f(P) = \langle 4, 0, 1 \rangle, \nabla g(P) = \langle 0, 0, 0 \rangle. \)
(d) \( \nabla f(P) = \langle -2, -6, 4 \rangle, \nabla g(P) = \langle 1, 3, -2 \rangle. \)
(e) \( \nabla f(P) = \langle 4, 0, 1 \rangle, \nabla g(P) = \langle 1, 3, -2 \rangle. \)

Correct answer: (e)
Which of the following is true?

(a) $D$ is closed and bounded, so by the extreme value theorem, $f$ has a maximum and a minimum on $D$.

(b) $D$ is not closed or bounded, so we can’t say anything.

(c) $D$ is not closed or bounded, but we can argue for geometric reasons that $f$ has a maximum and a minimum on $D$.

(d) $D$ is not closed or bounded, but we can argue for geometric reasons that $f$ has a maximum but not a minimum.

Correct answer: (d)