Is there a point on the graph \( z = \sqrt{x^2 + y^2} \) that's closest to the point \( D = (4,2,0) \)? Turment?

Recall the extreme value theorem:

Let \( f: D \to \mathbb{R} \) be continuous; \( D \) closed & bounded. Then \( f \) attains a maximum value at some point \( P \in D \), and either:

\( \cdot P \in \partial D = \text{boundary of } D \)

\( \cdot P \) is a critical point for \( f \).

Today: How can we find the maximum value for \( f \) over \( \partial D \)?

Example: Does \( f(x,y) = x^2 - y^2 \) have a maximum value on \( D = x^2 + y^2 \leq 4 \)? What is it?

- has max value by EVT, since \( D \) is closed & bounded.

- critical points: \( \nabla f = \langle 2x, -2y \rangle = \langle 0, 0 \rangle \) at \((0,0)\).

  \[ f(0,0) = 0. \]

- boundary points: \( \partial D = \{ x^2 + y^2 = 4 \} \) \( x \in [-2,2] \)

**Method 1**

\( g^2 = 4 - x^2 \)

= 1 on \( \partial D \), \( f(x,y) = x^2 - y^2 = 2x^2 - 4 \). \( x \in [-2,2] \)

So we need to find \( \max \) of \( g(x) = 2x^2 - 4 \) on \([-2,2]\)

- critical points: \( g'(x) = 4x = 0 \) \( \Rightarrow x = 0 \).

  \( g(0) = f(0, \pm 2) = -4 \)

- end points \( g(\pm 2) = f(\pm 2, 0) = (\pm 2)^2 - (0)^2 = 4 \)

So the maximum value of \( f \) is 4, at \((\pm 2, 0)\).

**Method 2**

\( f = 4 \)

\( g = 4 \) \( \pm \)-maxmin locations

let \( g(x,y) = x^2 + y^2 \)

\( \nabla g(x,y) = \langle 2x, 2y \rangle \)

\( \nabla f(x,y) = \langle 2x, -2y \rangle \)

max values at \((\pm 2, 0)\)

\( g = \langle \pm 4, 0 \rangle \)

\( \nabla f = \langle \pm 4, 0 \rangle \)
Min value at \((0, \pm 2)\) \[\nabla g = <0, \pm 4>\]
\[\nabla f = <0, \mp 4>\]

Exactly the locations where \(\nabla g, \nabla f\) point in the same direction (\(\pm 4\)).

**Theorem** (Lagrange multipliers) - discovered by Euler.

Assume \(f, g : \mathbb{R}^2 \to \mathbb{R}\) have continuous first order partial derivatives.

\(f\) is the maximum value of \(f\) on the level curve \(g(x, y) = k\) then either

- \(\nabla g(x_0, y_0) = <0, 0>\)

or
- \(\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)\) for some \(\lambda \in \mathbb{R}\).

**Note:** the theorem doesn't guarantee a maximum exists, it just tells us when it's possible a maximum does exist.

- But if \(g\) is continuous, \(fg = k\) is closed, so if we can show it's bounded, a maximum exists by EVT.

- If not, sometimes we can use a geometric/physical argument.

**Strategy:**
1. Check that \(fg = k\) is bounded
   - or use a geometric argument.
2. Check that \(\nabla g(x, y) \neq 0\) on \(g = k\)
3. Find all \(x, y, \lambda\) s.t.
   - \(\nabla f(x, y) = \lambda \nabla g(x, y)\) \(- two\) equations
   - \(g(x, y) = k\) \(- one\) more equation.
4. Calculate \(f(x, y) \vee (x, y)\) in (3).
5. Pick the largest.
Example: Find the max of \( f(x,y) = x^2 - y^2 \) on \( x^2 + y^2 = 4 \).

1. \( \{ g(x,y) = x^2 + y^2 = 4 \} \): boundary of \( D \) from before.
   - closed & bounded.

2. \( \nabla g(x,y) = <2x, 2y> \neq <0,0> \) on \( g=4 \).

3. 3 equations:
\[
\nabla f = \lambda \nabla g : \begin{cases} 
2x = \lambda 2x & \Rightarrow \lambda = 1 \text{ or } x = 0 \\
2y = \lambda 2y & \Rightarrow \lambda = -1 \text{ or } y = 0.
\end{cases}
\]
\[
\Rightarrow \lambda = 1 \text{ and } y = 0.
\]

Solutions are:
- \( x=0, y = \pm 2, \lambda = 1 \)
- \( y=0, x = \pm 2, \lambda = -1 \)

4. Evaluate:
\[
f(0, \pm 2) = -4
\]
\[
f(\pm 2, 0) = 4.
\]

5. Compare: \( \max \) is 4, \( \min \) is -4.

**Theorem** (Lagrange multipliers in 3 variables)

Assume \( f, g \) are functions of three variables with continuous first partial derivatives, then if \( f(x_0, y_0, z_0) \) is the max value of \( f \) on the level set \( g(x_0, y_0, z_0) = k \) then
\[
\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)
\]
for some \( \lambda \in \mathbb{R} \).

Note: Similar theorems hold for minima.

\[\square\] Example: Assume that \( f(x,y,z), g(x,y,z) \) have continuous first partial derivatives. Suppose that \( f \) achieves its maximum value on the level set \( g(x,y,z) = k \) at \( P \).

Which is not possible?
Example 15.4: Find the maximum volume of a box with surface area 60 m².

1. Function \( V(x,y,z) = xyz \)
2. Constraint: \( A(x,y,z) = 2xy + 2xz + 2yz = 6 \).
   
   But actually we have more constraints:
   
   \( x, y, z > 0 \).

So we're looking for the maximum of \( V \) over

\[
D = \{ (x,y,z) \mid A(x,y,z) = 6, \quad x > 0, \ y > 0, \ z > 0 \}
\]

II. Can we find a \( \lambda \) max?

System of equations:

\[
\begin{align*}
xy &= \lambda (2y + 2z) \\
xz &= \lambda (2x + 2z) \\
yz &= \lambda (2x + 2y) \\
2xy + 2xz + 2yz &= 6
\end{align*}
\]

\[
\Rightarrow \frac{1}{x} = \frac{1}{y} = \frac{1}{z} \Rightarrow x = y = z.
\]

\[
\Rightarrow \frac{1}{2x} = \frac{1}{2y} = \frac{1}{2z} \Rightarrow x = y = z.
\]

\[
\Rightarrow x = \pm 1
\]

But \( x > 0 \) \( \Rightarrow \) \( x = y = z = 1 \)

Volume is 1.

(Note: we never needed to solve for \( \lambda \))