1. Last time: Partial derivatives

E.g. Compute \( f_x(1,2) \) where \( f(x,y) = xe^{xy} \)

2. Look at the contour graph of \( f(x,t) \). [dark: negative light: positive]

At the point \( (x,t) = (\pi/2, 1.25) \) what can you say about the partial derivatives?

§ PARTIAL DIFFERENTIAL EQUATIONS (PDEs).

- In one variable we have "ordinary differential equations." (ODEs.)

Example: \( P(t) = \) population at time \( t \).

\[ P'(t) = cP(t) \]

\[ \Rightarrow P(t) = P_0 e^{ct} \]

Population at \( t = 0 \).

- In several variables we get equations involving partial derivatives.

E.g. if \( u(x,t) \) is the temperature in a rod at position \( x \) and time \( t \):

It turns out \( u \) satisfies the PDE

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \]  

Heat equation

Say temperature at time \( t = 0 \) is given by \( u(x,0) = \sin x \)

\[ \text{where } \frac{\partial^2 u}{\partial x^2} > 0, \text{ } u \text{ increases with time} \]

\[ \text{where } \frac{\partial^2 u}{\partial x^2} < 0, \text{ } u \text{ decreases.} \]

One Solution: \( u(x,t) = e^{-t} \sin x \)

Check: \( u_t = -e^{-t} \sin x \)

\( u_x = e^{-t} \cos x \)

\( u_{tt} = -e^{-t} \sin x \)
**Remark:** In this course, we don't need to find solutions to PDEs, we just need to verify whether a given function is a solution or not.

### 8. LINEAR APPROXIMATIONS

**Definition:** the linearization of \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) at \((a,b)\) is

\[
L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)
\]

**Remarks:**
- \( L \) is a linear function
- \( L(a,b) = f(a,b) \)
- \( \frac{\partial L}{\partial x}(a,b) = \frac{\partial f}{\partial x}(a,b) \quad \Rightarrow \quad \frac{\partial L}{\partial y}(a,b) = \frac{\partial f}{\partial y}(a,b) \)

\( \Rightarrow \) \( L \) is the linear function that's most like \( f \) near \((a,b)\).

\( \Rightarrow \) for \((x,y)\) near \((a,b)\) we can approximate \( f \) by \( L \):

\[
f(x,y) \approx L(x,y) \quad \text{"linear approximation"}
\]

**Example:**
\[
f(x,y) = xe^{xy} \quad \Rightarrow \text{linear approximation at } (1,0).
\]

\[
f_x(x,y) = e^{xy} + xy^2 e^{xy} \quad \Rightarrow f_x(1,0) = 1 + 0 = 1.
\]

\[
f_y(x,y) = x^2 e^{xy} \quad \Rightarrow f_y(1,0) = 1.
\]

\( \Rightarrow \)

\[
L(x,y) = 1 + 1(x-1) + 1(y-0)
\]

\[
= x + y
\]

Approximate \( f(x,y) \) at \((1.1, -0.1)\):

\[
f(1.1, -0.1) \approx 1.1 + (-0.1) = 1.
\]

(Actual answer: \( f(1.1, -0.1) = 0.98542 \ldots \))

The error in the linear approximation at \((a+\Delta x, b+\Delta y)\) is

\[
E(\Delta x, \Delta y) := f(a+\Delta x, b+\Delta y) - L(a+\Delta x, b+\Delta y).
\]

**Definition:** \( f \) is differentiable at \((a,b)\)

\[
\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{E(\Delta x, \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0.
\]
L is a good approximation to \( f \) near \((a,b)\).

If we zoom in on the graph of \( f \), we get the graph of \( L \).

**Example:** [Skip in lecture if too slow on time]

\[
f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}
\]

- \( f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0. \)
- Likewise \( f_y(0,0) = 0. \)

But \( f_x, f_y \) are not continuous:

For \((x,y) \neq (0,0), \quad f_x(x,y) = \frac{y}{x^2+y^2} - \frac{xy}{(x^2+y^2)^2} 2x = \frac{y^3-x^3y}{(x^2+y^2)^2}.

\[
\lim_{x \to 0} f_x(x,0) = 0
\]

\[
\lim_{y \to 0} f_x(0,y) = \lim_{y \to 0} \frac{y^3}{y} = \lim_{y \to 0} \frac{1}{y} = 0.
\]

The linear approximation would be \( f(x,y) \approx 0 \) near \((0,0).\)

But on the line \( x = y, \quad f(x,x) = \frac{1}{2}. \)

So this is a bad approximation.

§ TANGENT PLANES.

Let \( f: \mathbb{R}^2 \to \mathbb{R}; \) let \( c = f(a,b), \) so \((a,b,c)\) is a point on the surface \( \Gamma(f). \)
The tangent plane to $f$ at $(a,b,c)$ is the graph of the linear approximation to $f$ at $(a,b,c)$.

$$z = c + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

**Example:**

$$f(x,y) = 4 - x^2 - 2y^2$$

- paraboloid

$$\frac{\partial f}{\partial x}(x,y) = -2x$$

$$\Rightarrow \frac{\partial f}{\partial x}(1,1) = -2$$

$$\frac{\partial f}{\partial y}(x,y) = -4y$$

$$\Rightarrow \frac{\partial f}{\partial y}(1,1) = -4$$

$$c = 4 - 1 - 2 = 1$$

$$\Rightarrow \text{tangent plane is} \quad z = 1 - 2(x-1) - 4(y-1)$$

- i.e. $2x + 4y + z = 7$.

- **Slice at** $y = b = 1$

  $$\Rightarrow f(x,1) = 2 - x^2$$

  parabola.

  tangent line at $(1,1)$ has slope $\frac{\partial f}{\partial x}(1,1)$

- **Slice at** $x = a = 1$

  $$\Rightarrow f(1,y) = 3 - 2y^2$$

  parabola.

  tangent line at $(1,1)$ has slope $\frac{\partial f}{\partial y}(1,1)$

Tangent plane is the unique plane containing both these tangent lines and in fact all tangent lines to any curve in $\nabla(f)$ passing through $(a,b,c) = (1,1)$. 