Exercise 1: Pasch’s Axiom for Omega-Triangles, Part 2

In this exercise, we will prove the following:

**Theorem 1.** Let $PQ\Omega$ be an omega-triangle. Let $\ell$ be a line which passes through one of the sides, but not through a vertex $P$, $Q$ or $\Omega$. Then $\ell$ must pass through exactly one of the other two sides.

We will use Pasch’s Axiom for ordinary triangles, as well as Part 1 of Pasch’s Axiom for Omega-Triangles to prove the result. If you need to, remind yourself of these results.

a. First assume that $\ell$ passes through one of the infinite sides, say $\overrightarrow{P\Omega}$. Let $R$ be the point of intersection on $\overrightarrow{P\Omega}$, and draw the segment $QR$.

   (i) Draw this picture, including the points $P$, $Q$, $\Omega$, and $R$, the sides of the omega-triangle, and the segment $QR$. Do not draw $\ell$ yet. Notice that by assumption, $\ell$ must pass through some point $X$ which is interior to the triangle, but you do not know whether $X$ is in the interior of $\Delta PQR$ or of the new omega-triangle $QR\Omega$. (Why can’t $X$ be on the boundary?)

   (ii) Suppose $X$ is in the interior of $\Delta PQR$. Then what?

   (iii) Suppose $X$ is in the interior of $QR\Omega$. Then what?

b. Now assume $\ell$ intersects $PQ$ (but still does not pass through any vertices). Let $R$ be the point of intersection, and find the limiting parallel $\overrightarrow{R\Omega}$. Show that any line through $R$ and not through $\Omega$ (such as our line $\ell$!) must intersect either $\overrightarrow{P\Omega}$ or $\overrightarrow{Q\Omega}$.

c. Take stock. Convince yourself that you’ve finished the proof.

Exercise 2: Angle of parallelism

Recall the definition of the angle of parallelism: We started with a line $\ell$ and a point $P$ not on $\ell$. We found a line $m$ which was limiting parallel to $\ell$ through $P$; we drew the perpendicular line $p$ from $P$ to $\ell$, and we measured the angle that it formed with $m$ at $P$. This was the angle of parallelism of $\ell$ at $P$. We proved that it is always acute.

a. In this exercise, we will prove that the angle depends on only one thing: the distance $h$ between $P$ and the point $Q$ where the perpendicular line $p$ intersects the line $\ell$.

   More precisely, let $\ell'$ be another line. Let $Q'$ be a point on $\ell'$; let $p'$ be a line through $Q'$ perpendicular to $\ell'$, and let $P'$ be a point on $p'$ such that the distance between $P'$ and $Q'$ is $h$. Let $m'$ be limiting parallel to $\ell'$ at $P'$. Prove that the angle of parallelism in this case is congruent to the angle of parallelism of $\ell$ at $P$. (Use omega-triangle congruence.)

b. It follows that given any positive number $h$, we can find an angle $a(h)$, uniquely determined (up to congruence). This is called the angle of parallelism of $h$. We can also view $a(h)$ as a number between 0 and 90, the angle measure (e.g. in the Poincaré model).

c. Prove that if $h < h'$, then $a(h) > a(h')$ (i.e. the function $a$ is order-reversing).

   Hint: start with a line $\ell$, a point $Q$, and the perpendicular $p$ to $\ell$ through $Q$. Draw two points $P$ and $P'$ on $p$ corresponding to $h$ and $h'$, and draw the limiting parallels to $\ell$ at these points. Identify $a(h)$ and $a(h')$ in your picture. Use the Exterior Angle Theorem for Omega-Triangles.

d. Conclude that $a : \mathbb{R}_{>0} \rightarrow [0,90)$ is an injective function: that is, given $h_1 \neq h_2$, prove that $a(h_1) \neq a(h_2)$.