Exercise 1: A few more examples.

For any polygon $P$, we have

\[
defect(P) = 180(n-1) - \text{(angle sum of } P)\]
\[= 180(n-1) - \text{(angle sum of } \theta + \text{angle sum of } T)\]
\[= (180(n-2) - \text{angle sum of } \theta)\]
\[+ (180 - \text{angle sum of } T)\]
\[= \text{defect}(\theta) + \text{defect}(T)\]

\[
defect(P) = 180(n-2) - \text{(angle sum of } P)\]
\[= 180(n-2) - \text{(angle sum of } \theta + \text{angle sum of } T - 180)\]
\[= (180(n-2) - \text{angle sum of } \theta)\]
\[+ (180 - \text{angle sum of } T)\]
\[= \text{defect}(\theta) + \text{defect}(T)\]

\[
defect(P) = 180(n-3) - \text{angle sum of } P\]
\[= 180(n-3) - \text{(angle sum of } \theta + \text{angle sum of } T - 2\times180)\]
\[= (180(n-2) - \text{angle sum of } \theta)\]
\[+ (180 - \text{angle sum of } T)\]
\[= \text{defect}(\theta) + \text{defect}(T)\]

Induction Argument:

1. $n=1$, not coming to check.

2. Now suppose that we know that any polygon $Q$ which is completely triangulated into $n$ pieces, $T_1, \ldots, T_n$, has defect $\sum_{i=1}^{n} \text{defect}(T_i)$ (for fixed $n \geq 2$).

3. And suppose $P$ is triangulated into $T_1, \ldots, T_{n+1}$.

We know that at least one of the triangles, say $T_{n+1}$, has one or two free edges; call the union of the remaining triangles $Q$.

Then $(Q,T_{n+1}$) are as in one of our six examples, so

\[
defect(P) = \text{defect}(T_{n+1}) + \text{defect}(Q)\]

and by induction $\text{defect}(Q) = \sum_{i=1}^{n} \text{defect}(T_i)$. \qed
Practise with multiplying complex numbers: (Exercise 2)

(a) \( e^{i\phi} e^{i\psi} = (\cos \phi + i\sin \phi)(\cos \psi + i\sin \psi) \)

\[ = (\cos \phi \cos \psi - \sin \phi \sin \psi) + i(\sin \phi \cos \psi + \cos \phi \sin \psi) \]

\[ = \cos(\phi + \psi) + i\sin(\phi + \psi) \]

\[ = e^{i(\phi + \psi)} \]

(Note: this uses trig identities in its proof, but from now on you can use this fact to remember the trig formulas)

(b) \[ \frac{1}{2i} = \frac{1}{2i} \left( \frac{-2i}{-2i} \right) = \frac{-2i}{4} = -\frac{1}{2}i \]

\[ \frac{1+i}{1-i} = \frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{2i}{2} = i \]

\[ \frac{1}{2+y_i} = \frac{1}{2+yi} \frac{2-yi}{2-yi} = \frac{2-4i}{4+10} = \frac{2-4i}{20} = \frac{1}{10} - \frac{1}{5}i \]

Practise stereographic projection: (Ex 3)

\[ P = (x,y,0) \rightarrow (x,y,1) \]

\[ (x,y,1) \text{ is on the sphere, so } x^2 + y^2 + 1^2 = 1 \]

\[ \text{and it's on the line } \ell = \mathbb{P}N \quad (N=(0,0,1)) \]

\[ \{ tN + (1-t)P \} = \{ ((1-t)x, (1-t)y, 1) \} \]

So we must find \( t \) so

\[ (1-t)^2x^2 + (1-t)^2y^2 + 1^2 = 1 \]

i.e. \[(1-t)^2x^2 + (1-t)^2y^2 = 1-t^2 = (1-t)(1+t) \]

i.e. (since \( t \neq 1 \), we can divide by \( 1-t \))

\[ (1-t)(z^2 + y^2) = (1+t) \]

\[ t(z^2 + y^2 + 1) = z^2 + y^2 - 1 \]

\[ t = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \]

So \[ x = t = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \]

\[ y = (1-t)y = \frac{2y}{x^2 + y^2 + 1} \]