(a) Show that the symmetry group of a regular n-gon is finite, by defining an injective function $\Sym(P_n) \to S_n$.

Let $V_1, \ldots, V_n$ be the vertices of $P_n$.

Given $f \in \Sym(P_n)$, if permutes the vertices, giving a function

$$\Sym(P_n) \to \text{permutations of } V_1, \ldots, V_n = S_n$$

This function is injective: if $f, g$ give the same permutation, they agree on $V_1, V_2, V_3$, three non-collinear points.

$\Rightarrow$ the number of elements in $\Sym(P_n)$ is $\leq$ the number of elements in $S_n$ which is $n!$.

(b) Show that $\Sym(P_n) = D_n$.

Since $\Sym(P_n)$ is a finite planar symmetry group, we can identify it by finding the smallest rotation $R_{\alpha, x}$ and by determining whether it has any reflections.

we already know that it does have reflections (across angle bisectors or perpendicular bisectors of sides).

Note that the smallest rotation will be one of maximal order.

Since a rotation shifts the vertices cyclically and there are $n$ vertices, the order of a rotation is at most $n$.

So if we can find a rotation of order $n$, we will know that there are exactly $n$ rotations in $\Sym(P_n)$, and hence that $\Sym(P_n) = D_n$.

Consider the adjacent vertices $V_m, V_i, V_k$.

- Let $l_1$ be the angle bisector of $\angle V_mV_iV_k$.
- Let $l_2$ be the perpendicular bisector of side $V_iV_k$.
- Since $V_m, V_i, V_k$ are not collinear, $l_1$ and $l_2$ are not parallel, and they intersect at a point $O$, forming an angle $\phi$.

One can see that $\Delta V_iOV_k$ has angle $2\phi$ at $V_i$, and that for any $i$, $\Delta V_iOV_{i+1}$ is congruent (e.g. by ASA).

$\Rightarrow n(2\phi) = 360, \quad \Rightarrow 2\phi = 360/n$. 
and \( f_1 \circ f_2 = \text{Rot}_{0,2\theta} \) has order \( n \).

d. \ Symm(P_n) = D_n \quad \text{as claimed.} \quad \square

(c) Draw a figure whose symmetry group is either cyclic, not dihedral.
Exercise 3. Let \( l \) be a hypothetical line and let \( \Re \) be reflection across \( l \).

Suppose \( \mu m \) is limiting parallel to \( \mu \) and \( m \) is limiting parallel to \( l \).

Prove that \( \Re(\mu m) \) is limiting parallel to \( \mu \) as well.

Choose \( P \in \mu m; \) so \( \mu m \) is limiting parallel to \( \mu \) at \( P \).

Let \( \mu m' = \Re(\mu m); \) \( P' = \Re(P) \in \mu m' \).

We claim \( \mu m' \) is limiting parallel to \( \mu \) at \( P' \).

\( \mu m' \parallel \mu \Rightarrow \Re(\mu m') \parallel \Re(\mu) = \mu \) i.e. \( \mu m' \parallel \mu \).

Since \( P' = \Re(P); \) \( \overrightarrow{PP'} \perp \mu \); call the intersection point \( Q \).

We need to show that any ray \( \overrightarrow{P'X} \) interior to \( \angle QPR' \)
intersects \( \mu \).

Consider instead \( \Re(\overrightarrow{P'X}) = \overrightarrow{P'\Re(X)} \), interior to \( \angle QPR \).

Since \( \mu m \) is limiting parallel to \( \mu \) at \( P \), \( \overrightarrow{P\Re(X)} \) intersects \( \mu \) at some point \( T \).

\( \Rightarrow \Re(\overrightarrow{P\Re(X)}) = \overrightarrow{P'X} \) intersects \( \Re(\mu) = \mu \) at \( \Re(T) = T \).

Exercise 2. Let \( l \) and \( \mu m \) be limiting parallel.

Prove that they have no common perpendiculars.

Exercise 8. Let \( n \) be a second and \( \mu m \) be limiting parallel.

Suppose towards a contradiction, \( n \) is limiting parallel to \( \mu m \) at \( P \) with angle of parallellism \( \alpha = 90 \) \( \parallel \).

\( \parallel \)