Ex. 1: Suppose $f, g$ are isometries which agree on three non-collinear points $A, B, C$. Show that $f = g$.

Consider $f^{-1} g$, then

$\begin{align*}
  f^{-1} g(A) &= f^{-1}(f(A)) = A \\
  f^{-1} g(B) &= f^{-1}(f(B)) = B \\
  f^{-1} g(C) &= f^{-1}(f(C)) = C
\end{align*}$

Since $f^{-1} g$ is an isometry with three non-collinear fixed points, we have

\[ f^{-1} g = \text{id} \]

So

\[ f = g \]

Ex. 2: Prove that an isometry preserves circles.

Let $c = \{ x \mid d(x, c) = r \}$

$f(c) = \{ f(x) \mid d(f(x), f(c)) = r \} = \{ f(x) \mid f(0)f(x) = r \}

$because $f(c) = \{ f(x) \mid f(0)f(x) = 0 \}$

So $f(c)$ is the circle with centre $f(c)$ and radius $r$.

Ex. 3: (a) Define what it means for a set $S$ to be fixed or invariant under $r$.

- $S$ is fixed if $\forall P \in S$, $r(P) = P$
- $S$ is invariant if $\forall P \in S$, $r(P) \in S$

(b) Prove that the invariant lines of $r_2$ are exactly $l$ and the lines perpendicular to $l$.

- $l$ is fixed, so $l$ is invariant
- We need to prove that $\forall n \not\parallel l$ is invariant $\iff \exists \perp l$
r(um) c um,

take P e um, P \notin l, then r(P) \neq l, but by

assumption r(P) e um,

so um = Pr(P), and by the result in class,

l is the perpendicular bisector of Pr(P).

in particular um \perp l.

if um \perp l and P e um; we want to show r(P) e um.

if P \notin l, r(P) = P e um.

if P \notin l, we know l is the perpendicular bisector

of Pr(P), so r(P) is on the line m.

Ex. 4. (a). \[ T = \mathbf{e}_1 \circ \mathbf{e}_2 \]

with displacement vectors, \[ \mathbf{u} = (u_1, u_2) \]

\[ T^{-1} = \mathbf{e}_2 \circ \mathbf{e}_1 \]

Method 1: in a coordinate system \( v = (v_1, v_2) \), \( T \) is translation vector \( \mathbf{v} \).

\[ T(x, y) = (x, y) + (v_1, v_2) \]

let \( S(x, y) = (x, y) - (v_1, v_2) \) - by theorem, it is a translation

with translation vector \( -\mathbf{v} \).

It is easy to see that \( T \circ S = \text{id} \), so \( T = \text{id} \) or \( S = T^{-1} \).

But we also know that \( T^{-1} = (\mathbf{e}_1 \circ \mathbf{e}_2)^{-1} = \mathbf{e}_1^{-1} \circ \mathbf{e}_2^{-1} \]

\[ = \mathbf{e}_2 \circ \mathbf{e}_1 \]

Method 2: As above, \( T^{-1} = \mathbf{e}_2 \circ \mathbf{e}_1 \), and since \( \mathbf{e}_1 / \mathbf{e}_2 \) (or \( \mathbf{e}_1 = \mathbf{e}_2 \))

this is a translation.

To find the translation vector, take any point \( A \) and

look at the vector \( A + T^{-1}(A) = \mathbf{v} \)

Choose \( A = T(B) \) for some point \( B \).

so \( \mathbf{v} = T(B)B \).

But \( BT(B) = \mathbf{v} \), so

\[ T(B)B = -\mathbf{v} \]
Let $T_1, T_2$ be translations with displacement vectors $\vec{v}_1, \vec{v}_2$.

What is $T_1 \circ T_2$?

In coordinates, $\vec{v}_1 = (a_1, a_2), \vec{v}_2 = (b_1, b_2)$.

\[
\begin{align*}
T_1(x, y) &= (x, y) + (a_1, a_2) = (x + a_1, y + a_2) \\
T_2(x, y) &= (x + b_1, y + b_2)
\end{align*}
\]

$\Rightarrow T_1 \circ T_2 (x, y) = (x + a_1 + b_1, y + a_2 + b_2)

= (x, y) + (a_1 + b_1, a_2 + b_2)$

By theorem, this is a translation with translation vector $\vec{v} = (a_1 + b_1, a_2 + b_2) = \vec{v}_1 + \vec{v}_2$.

(c) Show that composition of translations commutes.

Is this true for stretches?

By (b), $T_1 \circ T_2 = \text{translation by } \vec{v}_1 + \vec{v}_2$

$T_1 \circ T_2 = \text{translation by } \vec{v}_2 + \vec{v}_1$

But $\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$, so $T_1 \circ T_2 = T_2 \circ T_1$.

On the other hand, by part (a), we see that for $l_1 / l_2$, $r_{l_2} \circ r_{l_1}$ and $r_{l_1} \circ r_{l_2}$ give translations in opposite directions, so $r_{l_1} \circ r_{l_2} \neq r_{l_2} \circ r_{l_1}$.

(d) Does the set of translations form a group?

Yes.

By (b), the composition of two translations is a translation.

Composition is always associative.

By definition, $id = r_{l_1} \circ r_{l_2}$ is a translation.

By (a), the inverse of a translation is again a translation.