Exercise 1. \( d_p(P, Q) = |\ln\left( \frac{PS}{PR} \cdot \frac{QR}{QS} \right)| \)

(a) Draw a picture showing R and S.

(b) Show \( d_p(R, Q) = 0 \iff P = Q. \) (Hint: compare the fractions \( \frac{PS}{QS}, \frac{PR}{QR}. \) What happens if \( PS < QS? \)

\[ \leq \] if \( P = Q, \quad \frac{PS}{PR} \cdot \frac{QR}{QS} = 1 \]

\[ \Rightarrow \quad d_p(P, Q) = |\ln(1)| = |0| = 0. \]

\[ \Rightarrow \] if \( d_p(C, Q) = 0, \quad \ln\left( \frac{PS}{PR} \cdot \frac{QR}{QS} \right) = 0 \]

\[ \Rightarrow \quad \frac{PS}{PR} \cdot \frac{QR}{QS} = 1. \]

\[ \Rightarrow \quad \frac{PS}{QS} = \frac{PR}{QR}. \]

If \( P \neq Q, \) then \( PS \neq QS. \)

WLOG assume \( PS < QS. \)

Then \( PR > QR, \) by inspection.

But then \( \frac{PS}{QS} < 1, \quad \frac{PR}{QR} > 1. \)

\( \Rightarrow \) \( P = Q. \)

(c) If \( Q = 0 = \) centre of unit circle, then simplify the formula.

\[ \frac{QR}{QS} = 1 \]

\( \cdot \) if \( PS = a, \) \( PR = 2 - a. \)

\[ \Rightarrow \quad d_p(P, Q) = |\ln\left( \frac{a}{2-a} \right)|. \]
Exercise 2:

(a) Draw the Klein disk, with a line \( \ell \), a point \( P \) not on \( \ell \), and two limiting parallels \( m, m' \) to \( \ell \) at \( P \).

(b) Draw the perpendicular line from \( P \) to \( \ell \). Label the angle of parallelism.

(c) Prove that there is no Klein line perpendicular to \( \ell \) and \( m, m' \).

A Klein line perpendicular to both \( \ell \) and \( m, m' \) must extend to a Euclidean line which passes through both \( \text{Pole}(\ell) \) and \( \text{Pole}(m, m') \).

But this Euclidean line is the tangent at \( A \), which does not intersect the interior of the disk, so it does not correspond to a Klein line.

* can also use: if \( n \) is perpendicular to \( \ell \) at \( Q \) and \( m, m' \) at \( P \), then \( n \) is a limiting parallel to \( \ell \) at \( P \) with angle of parallelism = 90°
Exercise 3:
(a) Prove that a bijection function must have a unique inverse.

Let \( f: S \rightarrow S \) be a bijection function.

Given any \( s \in S \), surjectivity of \( f \) implies \( \exists t \in S \) s.t. \( f(t) = s \).

Injectivity of \( f \) implies \( t \) is unique.

Define \( g(s) := t \); we claim \( g = f^{-1} \).

1. \( f \circ g = \text{id} \):
   - Take \( s \in S \) as above, \( t \) s.t. \( f(t) = s \).
   - So \( g(s) = t \).
   - \( f \circ g(s) = f(t) = s \) \( \forall s \in S \).

2. \( g \circ f = \text{id} \):
   - Take \( x \in S \), and let \( s = f(x) \).
   - \( g \circ f(x) = g(s) = \text{unique element } t \) s.t. \( f(t) = s \).
   - \( f(x) = s \implies \text{by uniqueness } t = x \).
   - \( g \circ f(x) = x \) \( \forall x \in S \).

Uniqueness of \( g \): If \( f \) has another inverse, \( g' \), then
\[
(f \circ g) = f \circ g' \implies \forall s \in S, f(g(s)) = f(g'(s))
\]
(since \( f \) is injective) \( g(s) = g'(s) \)
\[\implies g = g'.\]

(b) If \( f, g \) are invertible, check that \( h = f \circ g \) is invertible with \( h^{-1} = g^{-1} \circ f^{-1} \).

1. \( h \circ (g^{-1} \circ f^{-1}) = \text{id} \):
   - \( \forall s \in S, h \circ (g^{-1} \circ f^{-1})(s) = f(g(g^{-1}(f^{-1}(s)))) = f(f^{-1}(s)) = s. \]

2. \( (g^{-1} \circ f^{-1}) \circ h(s) = \text{id} \):
   - \( \forall s \in S, (g^{-1} \circ f^{-1}) \circ h(s) = g^{-1}(f^{-1}(f(g(s)))) = g^{-1}(g(s)) = s. \)
(c) Prove that if \( f \) is an isometry, then \( f^{-1} \) is also an isometry.

we need to show that \( \forall \ A, B, \ f^{-1}(A)f^{-1}(B) = AB \).

Applying the isometry \( f \) to \( X = f^{-1}(A), Y = f^{-1}(B) \)

\[ f(X)f(Y) = XY \]

But \( f(X) = f(f^{-1}(A)) = A \); \( f(Y) = f(f^{-1}(B)) = B \).

so this gives \( AB = f^{-1}(A)f^{-1}(B) \) as required.

(d) Prove that if \( f \) and \( g \) are isometries, \( fog \) is an isometry.

\[ f: \mathbb{R}^2 \to \mathbb{R}^2, \ g: \mathbb{R}^2 \to \mathbb{R}^2 \]

\[ \Rightarrow fog : \mathbb{R}^2 \to \mathbb{R}^2 \]

we need to show that \( \forall \ A, B, (fog)(A)(fog)(B) = AB \).

\[ (fog)(A)(fog)(B) = f(g(A))g(B) \]

\[ = g(A)g(B) \quad \text{since} \ f \ \text{is an isometry} \]

\[ = AB \quad \text{since} \ g \ \text{is an isometry} \]

(e) Prove that the set of isometries forms a group.

By part (d), composition gives a map

\[ \cdot : G \times G \to G \quad (\text{where } G = \{ f : \mathbb{R}^2 \to \mathbb{R}^2 \} \quad \text{if is an isometry}) \]

It is easy to see that \( \text{id} \in G \),

\[ f \circ \text{id} = f, \ \text{id} \circ f = f \quad \forall f \in G \]

so \( G \) has a unit.

associativity follows from associativity of composition.

By part (c), \( f \in G \Rightarrow f \) has an inverse \( f^{-1} \in G \).