Exercise 1.
(a) Write down the definition of a group. [5 points]

A group $G$ consists of a set of elements and a binary operation $\cdot$ that relates two elements of the group to a third.

The axioms for a group are:

(A1) For all elements $x$ and $y$ we have $x \cdot y \in G$.

(A2) The binary operation is associative. $\forall x, y, z \in G$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(A3) There is a special element $e \in G$ s.t. $x \cdot e = x \quad \forall x \in G$.

The element $e$ is called the identity of $G$.

(A4) Given $x \in G$, there is an element $x^{-1} \in G$ s.t. $x \cdot x^{-1} = e$.

The element $x^{-1}$ is called the inverse to $x$.

(b) Let $G$ be the set of invertible functions $\mathbb{R} \rightarrow \mathbb{R}$, with $\circ$ given by composition of functions. [10 points]

Prove that this is a model for the above axiomatic system.

(i) Associativity:

$$(f \circ (g \circ h))(x) = (f \circ g)(h(x)) = f(g(h(x)))$$

$$(f \circ (g \circ h))(x) = (f\circ g)(h(x)) = f(g(h(x))) \quad \forall x \in \mathbb{R}$$

$$\Rightarrow (f \circ g) \circ h = f \circ (g \circ h) \text{ as functions } \mathbb{R} \rightarrow \mathbb{R}.$$ 

(ii) Identity:

Let $e = \text{id}_\mathbb{R} : \mathbb{R} \rightarrow \mathbb{R}$. It is an invertible function,

$$x \mapsto x$$

and if $f \in G$ is any element,

$$(f \circ e)(x) = f(e(x)) = f(x) \quad \forall x \in \mathbb{R}$$

$$\Rightarrow f \circ e = f \text{ as required}$$

(iii) Inverses:

Each element $f \in G$ has an inverse $f^{-1} \in G$ by definition of $G$. 

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iv: Closure under $\circ$ (Axiom A1)

If $f, g \in G$, it is clear that $f \circ g \in G$. But we need to check that $f \circ g$ has an inverse.

Note that $f$ and $g$ each have inverses $f^{-1}$ and $g^{-1}$.

**Claim:** $g^{-1} \circ f^{-1}$ is an inverse for $f \circ g$

$$(g^{-1} \circ f^{-1}) \circ (f \circ g) = ((g^{-1} \circ f^{-1}) \circ f) \circ g$$

by associativity

$$= (g^{-1} \circ f^{-1}) \circ g$$

by associativity

$$= (g^{-1} \circ e) \circ g$$

since $f^{-1} \circ f = e$

$$= g^{-1} \circ g$$

since $g^{-1} \circ e = g^{-1}$

$$= e$$

Likewise, $(f \circ g) \circ (g^{-1} \circ f^{-1}) = e$.

**Exercise 2:**

(a) Let $G$ be a group, and suppose $x, y, z \in G$. Show that $x = y$.

$x \cdot z = y \cdot z$. Show that $x = y$.

$b \implies x \cdot z = y \cdot z^\prime \implies (x \cdot z) \cdot z^{-1} = (y \cdot z^\prime) \cdot z^{-1}$

But $(x \cdot z) \cdot z^{-1} = x(z \cdot z^{-1})$ by associativity

$$= x \cdot e$$

by property of inverses

$$= x$$

by property of identity

and likewise $(y \cdot z) \cdot z^{-1} = y$

$$\therefore x = y \text{ as claimed.}$

(b) If $x \in G$, $x \cdot e = e \cdot x$.

By (a), it suffices to show that $(x \cdot e) \cdot x^{-1} = (e \cdot x) \cdot x^{-1}$.

But $(x \cdot e) \cdot x^{-1} = x \cdot x^{-1}$ by property of $e$

$$= e$$

and $(e \cdot x) \cdot x^{-1} = e \cdot (x \cdot x^{-1})$ by associativity

$$= e \cdot e$$

by property of inverses

$$= e$$

by property of $e$.

The claim holds.
(c) The identity element is unique. [5 points]

proof: Suppose \( \exists \) an element \( e' \in G \) s.t. \( e' = x \cdot y \quad \forall x \in G \).

we want to show that \( e' = e \).

\[ e \cdot e' = e' \text{ by definition of } e' \]

But also \( e \cdot e' = e' \cdot e \text{ by (b)} \)

\[ = e' \text{ by definition of } e. \]

\[ \therefore e = e' \text{ as claimed.} \]

Exercise 3: \( S, \mathcal{P}(S \times S) \) s.t.

1. if \((a, b) \in \mathcal{P}(S \times S), (b, a) \notin \mathcal{P}(S \times S)\)
2. if \((a, b), (b, c) \in \mathcal{P}(S \times S), (a, c) \in \mathcal{P}(S \times S)\).

(a) Let \( S = \{1, 2, 3, 4\}, \quad P_1 = \{ (1, 2), (2, 3), (1, 3) \} \) [5 points]

Is this a model for the system?

Yes. We can show that the axioms hold.

(1) if \((a, b) \in P\) we need to show \((b, a) \notin P\)

\[ (a, b) = (1, 2) \implies (2, 1) \notin P; \]

\[ (a, b) = (2, 3) \implies (3, 2) \notin P; \]

\[ (a, b) = (1, 3) \implies (3, 1) \in P. \]

So axiom 1 holds.

(2) we need to check all pairs \((a, b), (b, c) \in P_1\).

The only such pair is \((1, 2)\) and \((2, 3)\).

The axiom says that \((1, 3)\) should be in \(P_1\), and it is.

(b) Let \( S_2 = \mathbb{R}, \quad P_2 = \{ (x, y) \mid x < y \} \). Is this a model? [5 points]

Yes. We check the axioms again.

1. if \((a, b) \in P_2\), then \(a < b\), so \(b \notin a\), and hence \((b, a) \notin P_2\)

2. if \((a, b), (b, c) \in P_2\), then \(a < b, b < c\),

so \(a < c\), and hence \((a, c) \in P_2\).
(c) Use this to argue that the axiomatic system is not complete.

We can add a third axiom which is independent from the first two and also consistent with them:

- $$(A3)$$ there are only finitely many elements of $$(S)$$

* The model $$(S_1, P_1)$$ satisfies this axiom, but the model $$(S_2, P_2)$$ does not.