Math 402: Exam 3
Fall semester 2018

- Do not forget to write your name and netid on top of this page.
- No notes, books, calculators, or other exam aids are allowed. You may use a ruler and colored pens or pencils if you wish.
- Turn your cell phones off and put them away. No use of cell phones or other communication devices during the exam is allowed.
- Write your answers clearly and fully on the sheets provided. If you need additional paper, raise your hand.
- Do not tear pages off of this exam. Doing so will be considered cheating.
- The exam consists of 5 problems and 7 pages. Check that your exam is complete.
- You have 50 minutes to complete the exam.

Good luck!!

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<tr>
<td>Total possible</td>
<td>20</td>
<td>20</td>
<td>27</td>
<td>21</td>
<td>12</td>
<td>100</td>
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<tr>
<td>Your points</td>
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Problem 1: \((8 + 5 + 7 = \mathbb{X}\text{ Points})\) Let \(\ell\) be a line, and \(P\) a point not on \(\ell\).

1. Consider the two limiting parallels \(m_1\) and \(m_2\) to \(\ell\) through \(P\). Let \(n\) be a line which bisects the angle made by \(m_1\) and \(m_2\) at \(P\) and intersects \(\ell\) at some point \(Q\). Is it true that \(n\) is perpendicular to \(\ell\) at \(Q\)? Why or why not?

Solution 1: Let \(m'\) be the perpendicular from \(P\) to \(\ell\). We know it makes congruent angles with \(m_1\) and \(m_2\) (angle of parallelism), so it is the line bisecting the angle at \(P\) made by \(m_1\) and \(m_2\). i.e. it is \(n\).

Solution 2: By \(\triangle\) congruence, \(\angle Q_1QP \cong \angle Q_2QP = 90^\circ\).

2. Define the \textit{angle of parallelism} of \(\ell\) at \(P\).

It is the angle at \(P\) made by the limiting parallel \((m_1, \alpha, m_2)\) and the perpendicular \(n\) to \(\ell\).

3. Can the lines \(\ell\) and \(m_1\) form opposite sides of a Lambert quadrilateral? Why or why not?

No. Opposite rows of a Lambert quadrilateral always have a common perpendicular.

\(\Rightarrow\) they are ultra-parallel, not limiting parallel.
Problem 2: (16 + 4 = 20 Points)

1. Let $F$ be a figure in the Euclidean plane such that the symmetry group $\text{Sym}(F)$ has exactly five elements. For each of the following statements, say whether it is true or false, justifying your answers.

(a) $F$ has translational symmetry.

   No. if $T \in \text{Sym}(F)$ is a translation then so is $T_n = Tn \quad \forall \ n \geq 0$. \Rightarrow \text{Sym}(F)$ is infinite.

(b) $\text{Sym}(F)$ has exactly as many rotations as it does reflections.

   No. $\text{Sym}(F)$ has no reflections, because if it had it would have even size.

(c) $\text{Sym}(F)$ contains only rotations.

   Yes. See (b).

(d) $\text{Sym}(F)$ might contain a rotation by 180°, but we can’t tell for sure.

   No. $\text{Sym}F$ is generated by $\text{Rot}_{\frac{\pi}{2}}$ and contains only id, $\text{Rot}_{\pi}$, $\text{Rot}_{2\pi}$, $\text{Rot}_{3\pi}$, $\text{Rot}_{4\pi}$.

2. Draw a picture of a figure $F$ whose symmetry group has exactly five elements.
Problem 3: \((14 + 13 = 27 \text{ Points})\)

(a) We did not prove, but it is true, that the area of an omega triangle \(PQ\Omega\) is equal to \(k^2(180 - (\angle PQA + \angle QPA))\).

Now suppose that you are given three omega points, \(\Omega_1, \Omega_2, \Omega_3\), which define a figure in the hyperbolic plane. Prove that this figure has area \(k^2 \cdot 180^\circ\).

\[
\text{Area}(\Omega_2, \Omega_3, \Omega_3) = \sum_{i=1}^{4} \text{Area}(\Delta T_i) \quad \cup \quad \text{Ry above fact} \quad \text{Area}(\Delta) \\
= \sum_{i=1}^{3} k^2(180 - (\alpha_i + \beta_i)) \cup k^2 \text{ defect}(\Delta) \\
= 4 \cdot 180 k^2 - 3 \cdot 180 k^2 \quad = \quad 180 k^2.
\]
(b) An omega-quadrilateral is a figure in the hyperbolic plane with three real vertices and one omega point as a vertex. Let $ABC\Omega$ and $A'B'C'\Omega'$ be two omega-quadrilaterals satisfying the following congruences:

$$\angle CBA \cong \angle C'B'A'$$

$$\angle BAO \cong \angle B'A'O'$$

$$\angle CBA \cong \angle C'A'B'$$

$$\angle BCO \cong \angle B'C'O'$$

$$AB \cong A'B'$$

Prove that $BC \cong B'C'$.

Consider the $\omega$-triangles $AB\omega$ and $A'B'\omega'$.

By SA congruence, $AB\omega \cong A'B'\omega'$

$$\implies \angle A'B'\omega' \cong \angle A\omega$$

$\therefore \angle CBO \cong \angle C'O'B'\omega'$

Now by AA congruence for $\omega$-triangles,

$BC\omega \cong B'C'\omega'$

and so $BC \cong B'C'$.
Problem 4: \((2 + 3 + 5 + 4 + 7 = 21 \text{ Points})\) In Euclidean geometry, we have three infinite families of regular tilings of type \((n, k)\), where \(n = 3, 4, 6\). (For each choice of \(n\) there are infinitely many tilings, because we can choose the tile to have any side length \(\lambda > 0\). Here is a picture of two different tilings of type \((4,4)\).)

In hyperbolic geometry, we can form a tiling of type \((5,6)\), for example. Let \(P\) be the tile. Answer the following questions about \(P\) and about this tiling. (Give justification, wherever necessary. You should probably draw a picture.)

(a) How many sides does \(P\) have?
   * \(n = 5\).

(b) What is the angle measure of an interior angle of \(P\)?
   \[
   \frac{360}{5} = \frac{360}{60} = 60 \degree \quad \text{because 6 corners meet at a vertex.}
   \]

(c) If we divide \(P\) into congruent isosceles triangles \(T\) with a common vertex at the centre of \(P\), what are the angles (base and summit) in each of these isosceles triangles?
   \[
   \alpha = \frac{360}{5} = 72 \degree, \quad \beta = \frac{360}{60} = 60 \degree
   \]

(d) What is the defect of \(P\)?
   \[
   \text{Defect (P)} = 180(5-2) - \text{angle sum} = 180(3) - 5 \cdot 60 = 540 - 300 = 240 \degree
   \]

(e) Why is there a unique tiling of type \((5,6)\), not an infinite family?
   By AAA congruence, the side lengths of \(T\) is uniquely determined and hence the side length of \(P\).
Problem 5: \( (6 + 2 + 4 = 12 \text{ Points}) \)

1. Let \( z = x + iy \) be a non-zero complex number. Its inverse \( \frac{1}{z} \) is also a complex number, so it can be expressed in the form \( a + ib \). Find formulas for \( a \) and \( b \).

METHOD 1: \[ \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} \]

METHOD 2: \[ z \frac{1}{z} = 1 \Rightarrow (x + iy)(a + ib) = (ax - by) + i(ay + bx) = 1 + i0 \]

\( ax - by = 1 \)

\( -bx^2 - bx = 1 \Rightarrow b = \frac{-y}{x^2 + y^2} \)

\( ay + bx = 0 \)

\( a = \frac{x}{x^2 + y^2} \)

\( \text{Assume } y \neq 0, \text{ otherwise } \frac{1}{z} = \frac{1}{x} \Rightarrow a = \frac{1}{x} \) and \( b = 0 \).

2. (a) The following shows a line in hyperbolic geometry. We have used two models this semester to study hyperbolic geometry, and we have given a map \( F \) between them. What is the name of the model this picture is using?

\[ \text{Poincaré} \]

\( \text{Klein line} \)

(b) Draw, on the same picture, what this line looks like in the other model of hyperbolic geometry (after applying \( F \) or \( F^{-1} \)).