I still have no voice! Let’s do another worksheet!

In this worksheet, we’ll finish our discussion of rotations, and move on to the last (!) class of isometries, glide reflections.

Recall that a rotation is an isometry that is composed of two reflections, \( R = r_m \circ r_\ell \), where \( m \) and \( \ell \) are two lines which are not parallel. So far we have proven the following things about rotations:

- \( R \) defined as above has a unique fixed point, which is the intersection point \( O \) of the lines \( m \) and \( \ell \). The point \( O \) is called the centre of rotation.
- If \( f \) is any isometry which has a unique fixed point, then it must be a rotation.

Our main goal for our study of rotations is to prove the following:

**Theorem 1.** Suppose that \( R \neq \text{Id} \) is a rotation about a fixed point \( O \). Let \( A \neq O \) be any other point, and set \( \ell = \overrightarrow{OA} \). Then there is a unique line \( m \) passing through \( O \) such that \( R = r_m \circ r_\ell \).

Furthermore, if the angle \( \angle AOR(A) \) has measure \( \theta \), then for any other point \( B \neq O \), the angle \( \angle BOR(B) \) has measure \( \theta \) as well.

**Why do we care about this theorem?** It tells us that rotations (defined in this funny way as the composition of two reflections) really do behave by rotating things: in the theorem, \( R \) has centre of rotation \( O \), and angle of rotation \( \theta \).

Conversely, from the proof we will see that given a point \( O \) and an angle \( \theta \), we can construct the rotation \( R \) about \( O \) of angle \( \theta \) by choosing two lines which intersect at \( O \) and form an angle of \( \frac{\theta}{2} \).

Great! What an important theorem! How do we prove it?

We’re going to need this theorem, which you proved at the end of Monday’s worksheet.

**Theorem 2.** Let \( \ell, m, n \) be three lines intersecting at the point \( O \). Then the composition \( r_\ell \circ r_m \circ r_n \) is a reflection, across some line \( p \) which also passes through \( O \).

Furthermore, \( r_\ell \circ r_m \circ r_n = r_n \circ r_m \circ r_\ell \).

Now we can carry out the proof of Theorem 1 as follows:

**Exercise 1.**

1. Recall that we have chosen \( A \neq O \), and we defined \( \ell = \overrightarrow{OA} \). Now let \( m \) be the angle bisector of \( \angle AOR(A) \). Prove that

\[
R = r_m \circ r_\ell.
\]

**Hint:** Show that \( r_m \circ R \) fixes \( O \) and \( A \), so \( r_m \circ R \) is either \( \text{Id} \) or \( r_\ell \)...

2. So that proves the first part of the theorem; let’s move on to the second. First show that it’s enough to prove that

\[
\angle AOR(A) \cong \angle B'OR(B'),
\]

where \( B' \) is the point on \( \overrightarrow{OB} \) such that \( OA = OB' = OR(B') \).

3. Let \( t \) be the angle bisector of \( \angle R(B')OA \). What is \( r_t(A) \)?

4. Use the previous steps to show that \( A = r_t \circ r_m \circ r_\ell(B') \), and so \( R(B') = r_t(A) \).

5. Use Theorem 2 to show that \( B' = r_t \circ r_m \circ r_\ell(A) = r_t(R(A)) \).

6. Use these facts, together with the fact that \( r_t \) is an isometry, to show that \( \triangle AOR(A) \cong \triangle B'OR(B') \), and in particular, \( \angle AOR(A) \cong \angle B'OR(B') \).
Exercise 2. Does this proof work in neutral geometry?

Now that we know that a reflection is determined by its centre and angle of rotation, we can consider its coordinate form.

Exercise 3.
Let $\text{Rot}_\phi$ denote rotation counterclockwise about $O = (0,0)$ by angle $\phi$. Prove that
\[
\text{Rot}_\phi(x,y) = x \cos \phi - y \sin \phi, x \sin \phi + y \cos \phi.
\]
Hints: use polar coordinates $(x, y) = (r \cos \theta, r \sin \theta)$ and the trig identities:
\[
\cos (\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi; \quad \sin (\theta + \phi) = \ldots
\]

Now we know everything there is to know about rotations, so let’s move on.

Recall that we proved that every isometry is a composition of at most three reflections. If it’s two reflections, it’s either a translation or a rotation. What happens if it’s three reflections? We saw in Theorem 2 that at least in some cases, the composition of three reflections is a reflection again. But it’s not always true.

Definition 1. A glide reflection is an isometry that is made up of a reflection and a translation parallel to the line of reflection.

Notation 1.
- Let $\vec{AB}$ be a vector, and let $\ell$ be a line parallel to the $\vec{AB}$.
- Let $T_{AB}$ denote translation by $\vec{AB}$.
- Let $r_\ell$ denote reflection across $\ell$.
- Then the glide reflection corresponding to $\vec{AB}$ and $\ell$ is $G_{\ell,AB} = T_{AB} \circ r_\ell$.

Why $T_{AB} \circ r_\ell$ and not the other way around? Our first result on glide reflections is that it doesn’t matter: the two ways of composing are equal (when $\ell$ and $AB$ are parallel as above).

Theorem 3.
\[
T_{AB} \circ r_\ell = r_\ell \circ T_{AB}.
\]

Exercise 4. Choose $P$ a point not on $\ell$. Let $P' = r_\ell P$ and let $P'' = T_{AB} (P)$. Let $G = T_{AB} (P'')$. Our goal is to show that $G = r_\ell (P'')$.

1. Why are each of the following parallelograms: $PP''GP'$; $APP''B$; and $AP'GB$?
2. Draw a picture (you can assume that $A$ and $B$ lie on the line $\ell$). Mark in all of the right angles that you know.
3. Prove that $\ell$ is the perpendicular bisector of $P''G$. Conclude that $G = r_\ell (P'')$ as desired.

Theorem 4.
\[
G_{\ell,AB}^{-1} = G_{\ell,BA}
\]

Exercise 5.
Prove this theorem, using your result from Theorem 3.

Next time we will prove that any isometry which is a composition of three reflections is either a reflection or a glide reflection, and we will ask when each case arises.