MATH 402 Review for November 27–December 1

Topics: 3.5 (complex numbers; definition of stereographic projection; Möbius transformations); 7.8 (a bijection between the Klein disk and the Poincaré disk); 8.1 (geometric properties of Möbius transformations); overview of isometries in hyperbolic geometry (not in the textbook).

These were covered in lectures and on Homework 10. Some of the later material will be on an optional homework assignment before the midterm.

1. Review from before this week:

(a) Definition of the stereographic projection $\pi : S^2 \to \mathbb{C}$ and its inverse. Definition of the function $F : \text{Disk}_K \to \text{Disk}_F$ and its inverse $F'$.

(b) Definition of perpendicular lines in the Klein model. Definition of the distance functions $d_P$ and $d_K$.

2. Key things about stereographic projection:

(a) If $(X, Y, Z)$ and $(x, y)$ are related by stereographic projection (i.e. $(x, y) = \pi(X, Y, Z)$), then we have the following formulas (with proof):
\[
X = \frac{2x}{x^2 + y^2 + 1}, \quad Y = \frac{2y}{x^2 + y^2 + 1}, \quad \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}; \quad x = \frac{1}{1-Z} X, \quad y = \frac{1}{1-Z} Y.
\]

(b) Stereographic project sends circles on the sphere to circles or lines on the plane. We get a line exactly when the circle on the sphere contains the North pole $N = (0, 0, 1)$.

* It follows that stereographic projection preserves angles.

3. Key things about $F$ and $F'$:

(a) $F$ sends Klein lines to Poincaré lines (with proof).

(b) The pole of a Klein line $\ell$ is equal to the centre of the circle $c_F(\ell)$ of the corresponding Poincaré line $F(\ell)$ (with proof).

(c) Two Klein lines $\ell$ and $m$ are perpendicular if and only if $F(\ell)$ and $F(m)$ are perpendicular.

(d) If $P, Q$ are two Poincaré points, then $d_P(P, Q) = d_K(F'(P), F'(Q))$.

4. Key things about Möbius transformations

(a) Definition: $f(z) = \frac{az + b}{cz + d}$, for some $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

(b) Any conformal bijection $f : \mathbb{C} \to \mathbb{C}$ is a Möbius transformation.

(c) A Möbius transformation with at least three fixed points must be the identity. Thus, two Möbius transformations which agree on at least three points actually agree everywhere.

(d) Important example: cross ratios. Fix $z_1, z_2, z_3 \in \mathbb{C}$ distinct points. Then define
\[
f(z) = (z, z_1, z_2, z_3) = \frac{z-z_2}{z-z_3} \cdot \frac{z_1-z_3}{z_1-z_2}.
\]

* $f(z) \in \mathbb{R}$ if and only if $z$ lies on the unique circle or line containing $z_1, z_2, z_3$.

* If $g$ is a Möbius transformation, then $(z, z_1, z_2, z_3) = (g(z), g(z_1), g(z_2), g(z_3))$ for all $z \in \mathbb{C}$. 
If $z_1, z_2, z_3$ lie on the circle $c$, then $z^*$ is the inverse of $z$ with respect to that circle if and only if

$$(z^*, z_1, z_2, z_3) = (z, z_1, z_2, z_3).$$

(c) A Möbius transformation maps circles and lines to circles and lines. It preserves inverses with respect to circles.

5. **Key things about isometries of the Poincaré disk**

(a) $f$ is an orientation-preserving isometry of the Poincaré disk if and only if it is a Möbius transformation which sends points on the boundary of the unit disk to points on the boundary of the unit disk, and sends points inside the unit disk to points inside the unit disk.

(b) Every such function can be written in the form

$$f(z) = \frac{\beta z - \alpha}{\bar{\alpha}z - \alpha},$$

for complex numbers $\alpha$ and $\beta$ such that $|\alpha| < 1$ and $|\beta| = 1$.

6. **Key things about hyperbolic isometries** Every orientation-preserving hyperbolic isometry can be written as a composition of two reflections $r_\ell \circ r_m$. In each case, identify the fixed points, the fixed omega points, and the invariant lines (if any):

(a) $\ell = m$.

(b) $\ell$ and $m$ intersect at a point $P$.

(c) $\ell$ and $m$ are ultraparallel.

(d) $\ell$ and $m$ are limiting parallel.

**Practice Questions**

1. **Practice with $F$, $F'$**

Prove (using drawings and geometric reasoning, or using the formulas for $F'$) that a Poincaré point $P$ and the corresponding Klein point have the same coordinates if and only if $P = (0,0)$. Also prove that an omega point in the Poincaré model has the same coordinates as the corresponding omega point in the Klein model.

2. **Practice with Möbius transformations**

(a) Use the first part of 4(c) above to prove the second part: that is, assume that $f$, $g$ are two Möbius transformations which agree on three points. Prove that $f = g$. (Hint: look at $g^{-1} \circ f$.)

(b) For $f(z) = (z, z_1, z_2, z_3)$ as in 4(d), find the values $f(z_1), f(z_2), f(z_3)$. Given another set of three distinct complex numbers $(w_1, w_2, w_3)$, use cross ratios to define a Möbius transformation that takes $z_i$ to $w_i$ for $i = 1, 2, 3$. 
