Exercise 1. (a) Suppose that \( z = x + iy \) is a point of the complex plane corresponding to the point \( P = (X,Y,Z) \) of the unit sphere under stereographic projection. Prove that
\[
X = \frac{2x}{x^2 + y^2 + 1}, \quad Y = \frac{2y}{x^2 + y^2 + 1}, \quad \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}.
\]
(Hint: recall that the line \( \ell \) between two points \( A \) and \( B \) has the form \( \{ tA + (1 - t)B \mid t \in \mathbb{R} \} \).
(b) Conversely, show that \( x = \frac{1}{1-Z}X \) and \( y = \frac{1}{1-Z}Y \).

Exercise 2.
(a) Let \( f(z) = \frac{az - b}{cz - d} \) be a Möbius transformation, so \( ad - bc \neq 0 \). Prove that we can find different complex numbers \( a', b', c', d' \) so that
\[
f(z) = \frac{a'z - b'}{c'z - d'}
\]
and \( a'd' - b'c' = 1 \).
(b) Show that the set \( \mathcal{U} \) of Möbius transformations that preserve the unit disk is a group.

Exercise 3. Fix \( \alpha \) with \( |\alpha| < 1 \). In this exercise, we consider the Möbius transformation given by \( T(z) = \frac{2z - \alpha}{2z - \alpha} \), and show why it makes sense to think of this as translation along the line \( \ell \) passing through 0 and \( \alpha \).
(a) Show that \( T \) has exactly two fixed points, which are both on the boundary of the unit disk. In particular, it has no fixed point inside the Poincaré disk.
(b) Show that \( T \) preserves the line \( \ell = \{ t\alpha \mid t \in \mathbb{R} \} \). (Note that this is the Euclidean line through 0 and \( \alpha \), but also restricts to the diameter corresponding to the hyperbolic line through 0 and \( \alpha \).) Show that the fixed points from part (a) are the omega points of this line.
(c) Show that for any Poincaré point \( t\alpha \) on \( \ell \), the Poincaré distance from \( t\alpha \) to \( T(t\alpha) \) is always the same.

Exercise 4. In this exercise we will see that inversion with respect to the circle defining a Poincaré line is the same as the hyperbolic reflection across that line in the Poincaré model.
Let \( \ell \) be a Poincaré line. Define a map \( f \) on the Poincaré disk by \( f(P) = P_0 \), where \( P_0 \) is the inverse point to \( P \) with respect to the circle \( c_\ell \) on which \( \ell \) is defined.
(a) Use results on circle inversion to show that \( f \) maps the Poincaré disk to itself.
(b) Prove that \( f \) is an isometry of the Poincaré disk (i.e. prove that it preserves the Poincaré distance function).
(c) Finally, show that \( f \) must be a reflection about the line \( \ell \).

Remember that in addition to the points assigned to each question, you will receive up to five further points for neatness and organization.