Math 402 · Homework 9 Solutions

10 a) Prove that in a Saccheri quadrilateral, the summit is always larger than the base.

5 pts
Let \(ABCD\) be a Saccheri quadrilateral.

Let \(E, F\) be the midpoints of the base \(AB\) and summit \(CD\).

We know that \(EF\) is perpendicular to \(AB\),
and divides \(ABCD\) into two congruent quadrilaterals.

\(EB < FC\), \(AE < DF\) since in a Saccheri quadrilateral a side adjacent to the acute angle is longer than the opposite side.

\(\Rightarrow\) \(EB + AE < FC + DF\), i.e. \(AB < CD\) as required.

b) Let \(ABCD\) and \(A'B'C'D'\) be two Saccheri quadrilaterals.

Suppose the summits are congruent and the summit angles are congruent. Prove that the two quadrilaterals are congruent.

Suppose the sides are not congruent.

WLOG assume \(AD > A'D'\).

Choose \(A''\) on \(AD\) s.t. \(A''D = A'D'\).

Likewise, choose \(B''\) on \(FC\) s.t. \(B''C = B'C'\).

Claim: \(A''B''\) is perpendicular to both sides.

Consider the triangle \(A''DC\) by SAS \(\triangle A''DC \cong \triangle A'D'C'\).

In particular, the angles \(\angle A''CD \cong \angle A'C'D'\).

It follows that \(\angle A''CB'' \cong \angle A'C'B'\).

Then by SAS again, \(\triangle A''CB'' \cong \triangle A'C'B'\).

In particular, the angle \(\angle A''B''C \cong \angle A'B'C' = 90^\circ\).

Likewise, the angle \(\angle B''A''D\) is also 90°.

This proves the claim, which implies that \(A''B''A''\) is a rectangle.
So \( AD = A'D' \), \( BC = B'C' \).

The argument with similar triangles above shows that \( AB = A'B' \) so

(c) Suppose you knew that the bases are congruent and the summit angles are congruent. Prove that the quadrilaterals are congruent.

Assume towards a contradiction that

\[ AD > A'D' \]

Choose \( D'' \) and \( C'' \) on \( AD, \ BC \) s.t.

\[ AD'' = A'D', \quad BC'' = B'C' \]

Claim: \( \angle AD''C'' \cong \angle A'D'C' \)

By SAS, \( \triangle ABC'' \cong \triangle A'B'C' \)

In particular, \( \angle BAC'' \cong \angle B'A'C' \), and so

\[ \angle C''AD \cong \angle C'A'D' \]

Also \( AC'' = A'C' \).

Then SAS \( \triangle AC''D'' \cong \triangle A'C'D' \), and in particular

\[ \angle AD''C'' \cong \angle A'D'C' \] as claimed.

Since \( ABC''D'' \) is a Saccheri quadrilateral, it follows that

\[ \angle AD''C'' \cong \angle BCD'' \cong \angle ADC \cong \angle BCD. \]

Then \( D''C''CD \) is a quadrilateral with angle sum 360°.

So \( AD = A'D' \), \( BC = B'C' \).

The above argument with similar triangles proves that \( DC = D'C' \) too.
4) Let \( l \parallel m \) be parallel lines. Show that three points \( A, B, C \) on \( l \) cannot all have the same distance to \( m \).

Suppose they do. WLOG assume \( A \neq B \neq C \).

Then if we draw the perpendiculars from \( A, B, \) and \( C \) to \( m \), they all have the same length.

\( \Rightarrow A'B'B \) and \( B'C'C \) are squares.

\( \Rightarrow \) the summit angles 1 and 2 are both acute.

But we must have \( 1 + 2 = 180^\circ \).

(5) Let \( l \parallel m \) be unparallel.

In the following picture prove that the distance from \( P \) to \( l \)

is smaller than the distance from \( P \) to \( m \).

10pts.

\( \text{Thm 1: using angle of parallelism} \)

We know that \( a(h) < a(h') \) \( \Rightarrow h > h' \), so it suffices to show that \( a(h) < a(h') \).

- If \( a(h') < a(h) \), then \( \angle Q'P'P = 180 - a(h') > 180 - a(h) \).

\( \Rightarrow \) the angle sum of the quadrilateral \( QQ'P'P \) is

\[ 90 + 90 + a(h) + (180 - a(h')) > 360^\circ \]

\( \Rightarrow a(h') > a(h) \).
Method 2: Using hyperbolic quadrilaterals

Draw the perpendicular from $P$ on $n'$, intersecting $n'$ at $X$.

Note: $X'$ must be above $P'$ if it is below as in the picture, then $\Delta PXD'$ has a right angle at $X$ and an acute angle at $P'$.
(cause acute angles are always acute) 

So $X$ is above $P'$.

Now $QQ'XP$ is a hyperbolic quadrilateral with acute angles at $P$.

\[ PQ > XQ' > P'Q' \]

as required. \( \blacksquare \)
3) a) Prove that if a polygon is made up of subtriangles, the defect of the polygon is the sum of defects of the triangles.

Note: we can assume the triangles meet at common vertices — if not, we just subdivide more.

Given such a subdivision, a given triangle shares either 1, 2 or 3 sides with other triangles, and there are always triangles with only one or two sides in common with the other triangles.

\[ \text{e.g.} \]

By induction, it is enough to show that if we remove only one of these triangles at a time, the defect still exists.

Note: if \( P \) is a polygon with \( n \) sides,

\[ \text{Defect}(P) = 180(n-2) - (\text{angle sum of } P) \]

Case 1: \( T \) has two sides free, not collinear with adjacent sides.

\[ P = T \cup Q \quad \text{if } P \text{ is an } n \text{-gon, } Q \text{ is an } (n-1) \text{-gon} \]

\[ \text{Defect}(P) = 180(n-2) - (\text{angle sum of } P) \]

\[ = 180(n-2) - (\text{angle sum of } Q + \text{angle sum of } T) \]

\[ = 180(n-3) - (\text{angle sum of } Q) \]

\[ + 180 - (\text{angle sum of } T) \]

\[ = \text{Defect}(Q) + \text{Defect}(T) \]

Case 2: \( T \) has two sides free, one collinear with an adjacent side.

\[ T \quad \text{then } Q \text{ is still an } n \text{-gon.} \]

\[ \text{Defect}(P) = 180(n-2) - (\text{angle sum of } P) \]

\[ = 180(n-2) - (\text{angle sum of } Q + \text{angle sum of } T - 180) \]
\[
\text{Defect } Q + \text{Defect } T
\]

Case 3: T has two sides free, both collinear with adjacent sides

\[
\Rightarrow Q \text{ is an } (n+1)-\text{gon}
\]

\[
\text{Defect } (P) = 180(n-2) - (\text{angle sum of } P)
\]

\[
= 180(n-2) - \text{[angle sum of } Q) + \text{angle sum of } T
\]

\[
- 360
\]

\[
= 180(n-1) - \text{[angle sum of } Q)
\]

\[
+ 180 - \text{angle sum of } T
\]

\[
= \text{Defect } (Q) + \text{defect } (T).
\]

Case 4: T has one side free, no collinear with adjacent sides

\[
\Rightarrow Q \text{ is an } (n+1)-\text{gon}
\]

\[
\text{Defect } (P) = 180(n-2) - (\text{angle sum of } P)
\]

\[
= 180(n-2) - (\text{angle sum of } Q + \text{angle sum of } T
\]

\[
- 360
\]

\[
= 180(n-1) - \text{angle sum of } Q
\]

\[
+ 180 - \text{angle sum of } T
\]

\[
= \text{Defect } (Q) + \text{defect } (T).
\]

Two more cases for T with one side free:

1. Q is an \((n+2)\)-gon
2. Q is an \(n(n+3)\) gon
(b) Use this to show that if two triangles are equivalent, they have the same defect.

4pt3

$T$ is equivalent to $T'$ if they can be cut up into corresponding polygonal pieces.

Each of the polygons can be subdivided into triangles.

$T$ and $T'$ are equivalent $\iff$ they can be cut into subtriangles which can be matched in augmenting pairs $S_i t$ and $S_i t'$.

$S_i \equiv S_i'$, and in particular,

$\text{Defect } (S_i t') = \text{Defect } (S_i t)$

Now $\text{defect } (T) = \sum_i \text{Defect } (S_i t)$

$= \sum_i \text{Defect } (S_i t')$

$= \text{Defect } (T') \quad \Box$