(a) \( T = c_{l_1} \circ c_{l_2} \) translation with displacement vector \( \overrightarrow{u} \)

Prove that \( T^{-1} \) is also a translation

\( \overrightarrow{u} = T(A) - A \) for any point \( A \)

The displacement vector of \( T^{-1} \) is given by

\[
\overrightarrow{w} = T^{-1}(B) - B
\]

Taking \( B = T(A) \),

\[
\overrightarrow{w} = T^{-1}(T(A)) - T(A) = A - T(A) = T^{-1}(A) - A
\]

\[
= -\overrightarrow{u}
\]

(b) let \( T_1 \) and \( T_2 \) be translations with displacement vectors \( \overrightarrow{u}_1 \) and \( \overrightarrow{u}_2 \). What is \( T_1 \circ T_2 \)?

\[
T_1 \circ T_2 \left( x, y \right) = T_1 \left( T_2 \left( x, y \right) \right) = T_1 \left( x, y + \overrightarrow{u}_2 \right)= \left[ \left( x, y \right) + \overrightarrow{u}_2 \right] + \overrightarrow{u}_1
\]

\[
= \left( x, y \right) + \left( \overrightarrow{u}_1 + \overrightarrow{u}_2 \right)
\]

\[
= T_1 \circ T_2 \text{ is a translation with displacement vector } \overrightarrow{u}_1 + \overrightarrow{u}_2
\]

(c) Show that \( T_1 \circ T_2 = T_2 \circ T_1 \). Is it also true that composition of reflections commutes?

The same argument shows that \( T_2 \circ T_1 \) is a translation with displacement vector \( \overrightarrow{u}_1 + \overrightarrow{u}_2 \). So \( T_1 \circ T_2 = T_2 \circ T_1 \).

But if we look at a reflection \( r_o \alpha \) and suppose \( \alpha = 30^\circ \)

Then \( r_o \alpha \) is rotation about \( O \) by some angle \( \phi \)

Since it is rotation by angle \( -\phi \), it is different

unless \( \phi = \phi \pmod{360^\circ} \)
(a) Does the set of translations form a group?

Yes - it is closed under \( \circ \) by (6)

- \( \circ \) is associative by HW1
- \( \circ \) has an identity element \( \text{Id} = 0 \in \mathbb{R}^2 \)
- elements have inverses which are also translations by (a)

[2] Let \( T \neq \text{Id} \) be a translation.

Prove that \( l \) is an invariant line for \( T \) if \( l \) is parallel to the displacement vector \( \overrightarrow{S} \) of \( T \).

Let \( l \) be a line. Choose a representation \( \overrightarrow{AB} \) of \( l \) where \( A \in \mathbb{R}^2 \).

Take \( P \in \mathbb{R}^2 \). We know \( \overrightarrow{AP} + \overrightarrow{PD} = \overrightarrow{AD} \) of \( \mathbb{R}^2 \).

\[ \overrightarrow{AB} \parallel \overrightarrow{PD} \]

\[ \Rightarrow l \text{ invariant } \rightarrow T(P)l \Rightarrow \overrightarrow{TPD} = \overrightarrow{L} \]

\[ S = \overrightarrow{L} \parallel \overrightarrow{AB} \text{ and hence to } \overrightarrow{S} \]

\[ \overrightarrow{PDL} \parallel \overrightarrow{AB} \text{ and passes through } P \]

Pascal's theorem implies \( \overrightarrow{TPD} = \overrightarrow{L} \)

\[ T(P)l \Rightarrow l \]

Since \( P \) was arbitrary, \( l \) is invariant

[3] a) Let \( C = \langle x, y \rangle \), let \( T = T(x, y) \)

Prove that \( T^{-1} \circ \text{Rot}_\phi = T^{-1} \) is rotation about \( C \) by angle \( \phi \).

Let \( l \) be the line through \( (C, 0) \) perpendicular to \( \overrightarrow{C(0, y)} \)

Let \( l' \) be the perpendicular bisector of the segment \( \overrightarrow{CD} \)

Let \( n \) be the angle bisector of the angle made by \( l \) and \( \text{Rot}_\phi(C) \)

Then:

1. \( T = \text{Rot}_\phi(C) \)
2. \( \text{Rot}_\phi = \text{Rot}_\phi(C) \)
(a) Give an expression for reflection \( r_\phi \) across \( \ell \).

- Let \( r_x \) = reflection across \( x \)-axis, \( (y) \mapsto (x+y) \).
- Reflection across a line \( \ell \) through \( O \) can be written as follows:
  1. Rotate so the line coincides with the \( x \)-axis (\( \text{Rot}_\theta \), say).
  2. Reflect across the \( x \)-axis \( (r_x) \).
  3. Rotate back \( \text{Rot}_\phi \).

Case: If \( P \in \ell \), \( \text{Rot}_\phi \circ r_x \circ \text{Rot}_\phi^{-1} (P) = \text{Rot}_\phi \circ (r_x \circ \text{Rot}_\phi^{-1}) (P) = P \) because \( \text{Rot}_\phi (P) \) is on the \( x \)-axis.

So \( \ell \) is a fixed line, and
\[
\text{Rot}_\phi \circ r_x \circ \text{Rot}_\phi = r_{\phi/2},
\]
if \( H = \text{Id} \), then \( r_x = \text{Id} \) so
\[
\text{Rot}_\phi \circ r_x \circ \text{Rot}_\phi = r_{\phi/2}.
\]
Finally let m be any ray through O. Fix a point P on m.
and let T_p be translation taking O to P.

So l = T_p(m) is a line through O.

Reflection across m can be broken into steps as follows:

1. Translate m to l (T_p)
2. Reflect across l (r_x)
3. Translate back to l (T_p)

Check if Q(m, T_p(Q)) ∈ l, so r_x ° T_p°(Q) = T_p°(Q)

. . . T_p° o r_x o T_p°(Q) = Q

T_p ° r_x ° T_p = \left\{ \begin{array}{ll}
\text{Id} & \text{if } T_p ° r_x ° T_p = \text{Id} \\
\text{id} & \text{otherwise}
\end{array} \right.

Continuing:

[\text{Id} = T_p ° r_x ° T_p°]

= T_p ° \text{Rot} - \phi ° r_x ° \text{Rot} - \phi ° T_p°

\[ \text{(4) a) Prove that } (\text{Rot}_\phi)^{-1} = \text{Rot}_{-\phi} \]

Write Rot_\phi = \text{smere} \text{ where } l,m \text{ intersect at } O \text{ forming angle } \Psi/4

then (Rot_\phi)^{-1} = ((\text{smere})^{-1} = r_x^{-1} ° r_m^{-1} = r_x ° r_m

m, l \text{ intersect at } O \text{ forming the angle with opposite orientation = 0 has measure } -\Psi/2

\[(\text{Rot}_\phi)^{-1} = \text{Rot}_{-\phi} \]

\[ \text{b) Prove that Rot}_\phi \circ \text{Rot}_{-\phi} \text{ is a rotation about } O \]

Write Rot_{-\phi} = \text{smere} \text{ for some line intersecting at } O \text{ with angle } \Psi/2

We saw that choosing m to be the angle bisector of the angle at
O made by m and Rot_\phi(m). We can write Rot_\phi = \phi° r_m

\[ \text{Rot}_\phi ° \text{Rot}_{-\phi} = (\text{Rot}_\phi ° r_m ° r_m ° r_m ° r_m) = r_m \circ r_m \]
l and n intersect at a acute angle $\phi/2 + \psi/2 = (\phi + \psi)/2$

$\Rightarrow$ Rot $\phi$ a Rot $\psi$ = Rot $\phi + \psi$

(c) Let A, B be two different points. Let $R_1, R_2$ be rotation about A and B by 180°.

What is $R_2 \circ R_1$?

\[
\begin{align*}
R_2(R_1(A)) &= R_2(A) = A + 2\vec{AB} \\
R_2(R_1(B)) &= B + 2\vec{AB} \\
R_2(R_1(C)) &= C + 2\vec{AB}
\end{align*}
\]

So $R_2 \circ R_1$ agrees with $T_{2\vec{AB}}$ on 3 non-collinear points.

\[\therefore R_2 \circ R_1 = T_{2\vec{AB}}\text{, translation by one vector } 2\vec{AB}\]

(d) $R$ = 3 rotations.

$R_0 = $ rotations about $O$.

Is $R$ a group? Is $R_0$ a group?

$R$ is not a group, because $R_1 \circ R_2$ need not be a rotation as in part (c).

$R_0$ is a group - closure under $\circ$ is part (B).

- associativity is from H.W.A.
- unit element $I_0 = R_0 \circ 0$ degrees
- inverses exist (and are in $R_0$) by part (c)