Exercise 1. Let $S$ be any set, and let $f, g : S \to S$ be any two functions. Recall that we say that $g$ is the inverse of $f$ (and write $f^{-1} := g$) if for every $s \in S$ we have

$$f(g(s)) = s; \quad g(f(s)) = s.$$ 

A function $f$ which has an inverse is called invertible.

a. A function $f : S \to S$ is called bijective if it is both injective (‘one-to-one’) and surjective (‘onto’). Prove that a bijective function $f$ must have a unique inverse.

b. Prove that if $f$ and $g$ are invertible with inverses $f^{-1}$, $g^{-1}$, and if $h = f \circ g$, then $h$ is invertible with $h^{-1} = g^{-1} \circ f^{-1}$.

c. In particular, we defined a transformation to be a bijection of the plane, so it follows immediately from the above that a transformation has an inverse. Recall that a transformation is called an isometry if it preserves length. Prove that if $f$ is an isometry, then its inverse is also an isometry.

d. Prove that if $f$ and $g$ are isometries, then $f \circ g$ is an isometry.

e. Combine the last two parts of the exercise, and use the fact that composition of functions is associative, to show that the set of isometries is a group. (You may need to review the definition of a group! Make sure you address each group axiom in your solution.)

Exercise 2. Prove the following theorem:

Theorem 1. Suppose that $f$ and $g$ are two isometries which agree on three non-collinear points $A, B, C$. Prove that $f(P) = g(P)$ for all points $P$.

Exercise 3. Prove that an isometry preserves circles: i.e. if $f$ is an isometry, and $c$ is a circle with radius $r$ and centre $O$, then $f$ maps $c$ to the circle $c'$ of radius $r$ and centre $f(O)$.

Exercise 4. A reflection $r$ is defined as an isometry which has two fixed points, and which is not the identity.

a. Prove that $r^2 = \text{Id}$. Thus, a reflection is its own inverse.

b. Define what it means for a set $S$ to be fixed by $r$. Define what it means for $S$ to be invariant under $r$.

c. Recall that we proved that the reflection fixes the entire line $\ell$ determined by these two points, and we denoted this reflection by $r = r_{\ell}$. Now prove that the invariant lines of $r_{\ell}$ are exactly the line $\ell$ and the lines $m$ which are perpendicular to $\ell$.

Remember that in addition to the points assigned to each question, you will receive up to five further points for neatness and organization.