a. (i) Prove that $m \angle AOB = 2 m \angle APB$.

Let $\alpha = m \angle AOB$, $\beta = m \angle APB$.

By Isosceles $\triangle$ Theorem, $\angle QPA = \angle PQA$.

Call $m \angle QPA = \gamma$.

Likewise $m \angle PBO = m \angle PBR = \gamma + \beta$.

So the angle sum of $\triangle OBP$ is $180 = \angle LBOP + 2\gamma + 2\beta$.

And the angle sum of $\triangle GAP$ is $180 = 2\gamma + (m \angle LBOP + \alpha)$.

$\alpha = 2\beta$ as claimed.

(ii) Prove that $m \angle AOB = 2 m \angle APB$.

By Isosceles $\triangle$ Theorem, $m \angle APB = m \angle PBQ = \gamma$ say.

So the angle sum of $\triangle PQO$ is $2\gamma + m \angle LBOP = 180$.

But also: $180 = m \angle LBOP + m \angle APQ$.

$\therefore m \angle LAOP = 2\gamma$ as claimed.

b) Let $ABCD$ be a quadrilateral inscribed in a circle.

Prove that the angle at $A$ and the angle at $C$ are supplementary.

If $BD$ is a diameter, both $\angle A$ and $\angle C$ are right angles, so they are supplementary.

If $BD$ is not a diameter, one of $A$ and $C$ must lie on the major arc. Assume it is $C$.

So $m \angle DCB = \frac{1}{2} m \angle DOB$.

By IAT we can label the angles at $B, A, D$ as above.

$180 = 2\alpha + \beta$, $180 = 2\beta + \gamma$

$= 360 = 2\alpha + 2\beta + \beta + \gamma = 2(\alpha + \beta) + 2(m \angle DCB)$.

$= 180 = m \angle BAD + m \angle DCB$ as required.
Complete the proof of the law of sines:

Prove that \( \frac{a}{\sin \angle A} = \alpha \) (\( \alpha = BD \))

\( A \)  is on the minor arc, \( \angle LA \) is not an inscribed angle.

But \( ABDX \) is an isosceles quadrilateral, and \( \angle LA = 180 - \angle LD \)

\( \Rightarrow \sin \angle LA = \sin \angle LD \)

And since \( BD \) is a diameter, \( m \angle BCD = 90 \), so

\( \sin \angle LA = \sin \angle LD = \frac{\alpha}{BD} \)

Rearranging, \( \frac{\alpha}{\sin \angle LA} = d \) as claimed.

2 (a) Let \( A, B \) be distinct points. Show that

\( \overrightarrow{AB} = fC \quad | \quad C = \overrightarrow{A} + t(\overrightarrow{B} - \overrightarrow{A}), \quad t \in [0, 1] \)

Suppose \( A = (a,b), \quad B = (c,d) \)

The line \( L \) through \( A \) and \( B \) has equation

\[ L = f(x, y) \quad | \quad (d-b)x + (a-c)y + bc - ad = 0 \]

[This is a line; comment plug in \((x, y) = (a, b), \quad (x, y) = (c, d)\).

and see the line passes through \( A \) and \( B \).

Let \( L^0 = \{ C | \quad C = \overrightarrow{A} + t(\overrightarrow{B} - \overrightarrow{A}), \quad t \in [0, 1] \} \)

If \( C \in L^0 \), \( C = ((1-t)a + c, (1-t)b + d) \)

Claim: \( C \in L^0 \):

\[ (d-b)[(1-t)a + c] + (a-c)[(1-t)b + d] + bc - ad = 0 \]

\( S_bL^0 C \).

Furthermore, \( C \in L^0 \Rightarrow d(A, C) = \sqrt{(-ta + c)^2 + (-tb + d)^2} \]

\[ \leq \sqrt{(-a + c)^2 + (-b + d)^2} = d(A, B) \]

and likewise \( d(C, B) \leq d(A, B) \)

So \( A + C \leq B \)

So \( L^0 \subseteq \overrightarrow{AB} \)

Finally, \( L^0 \) is a continuous line segment containing \( A \) and \( B \),

So \( \overline{AB} \subseteq L^0 \quad \Rightarrow \quad L^0 = AB \)
b) Show that the midpoint M of the segment \( \overline{AB} \) satisfies \( M^2 = \frac{1}{2}(\overline{A}^2 + \overline{B}^2) \)

Note that \( \frac{1}{2}(\overline{A} + \overline{B}) = \overline{M} \) and \( \frac{1}{2}(\overline{B} - \overline{A}) = \overline{M} \). Thus, \( M \in \overline{AB} \).

Also, \( d(A, M) = \sqrt{(a - \frac{1}{2}(a+c))^2 + (b - \frac{1}{2}(h+d))^2} \)
\[
= \sqrt{(\frac{1}{2}a - \frac{1}{2}c)^2 + (\frac{1}{2}b - \frac{1}{2}d)^2}
\]
\[
= \sqrt{(\frac{1}{2}c - \frac{1}{2}c)^2 + (\frac{1}{2}d - \frac{1}{2}d)^2}
\]
\[
= d(B, M).
\]

So \( M \) is indeed the midpoint.

3. a) Show that the line passing through the centre of a circle and the midpoint of a chord (which is not the diameter) is perpendicular to the chord.

\( \overline{AO} \cong \overline{BO} \) because they are radii of the same circle.

SSS congruence \( \Rightarrow \Delta AMO \cong \Delta BMO \)

So \( \angle AMO \cong \angle BMO \).

But also the angles are supplementary; therefore they must be right angles.

b) Show that the line from the centre of a circle to an outside point bisects the angle made by the two tangents from the point to the circle.

\( \overline{TP} \) is tangent to the circle at \( T \)

\( \Rightarrow \) it is perpendicular to \( \overline{OT} \)

Likewise \( \overline{TP} \) is perpendicular to \( \overline{OT'} \).

Furthermore, \( \overline{OT} \cong \overline{OT'} \) because they are radii.

So \( \Delta OTP \) and \( \Delta OTP' \) are right triangles with congruent hypotenuses and a congruent leg.

By Ex 2.1.11, \( \Delta OTP \cong \Delta OTP' \).

In particular \( \angle OPT \cong \angle OPT' \).
a) Prove that two distinct Poincaré lines $l, l'$ intersect at most once inside the Poincaré disk.

Suppose that $P, Q$ are distinct points in the intersection of $l, l'$ inside the disk.

i) If either $P$ or $Q$ is at $O$, then $l, l'$ are diameters of the Poincaré disk, and in particular they extend to Euclidean lines containing $P$ and $Q$.

But there is only one such Euclidean line, so $l = l'$.

ii) If neither $P$ nor $Q$ is at $O$, then theorem (a) tells us $l, l'$ circle or line through $P$ and $Q$ orthogonal to the unit circle.

So two distinct Poincaré lines can only intersect once.

b) Hyperbolic distance: $d_H(P, Q) = \ln \left| \frac{(PS)(QR)}{(PR)(QS)} \right|$

Draw a picture:

\[\text{(or } P, R, S \text{ can be swapped)}\]

c) Show that $d_H(P, Q) = 0 \iff P = Q$.

\[\ln(x) = 0 \iff x = 1\]

So $d_H(P, Q) = 0 \iff \frac{PS}{PR} \frac{QR}{QS} = 1$.

- If $P \neq Q$, $PR < QR, PS > QS$,

  \[\Rightarrow \frac{PS}{PR} \frac{QR}{QS} = \frac{PS}{PR} \frac{QR}{QS} < 1 \Rightarrow d_H(P, Q) \neq 0\]

- If $P = Q$, $\frac{PS}{PR} = 1, \frac{QR}{QS} = 1 \Rightarrow d_H(P, Q) = 0$.

d) If $Q = O$, simplify the formula for $d_H(P, O)$.

\[d_H(P, O) = \ln \left| \frac{(PS)(QR)}{(PR)(QS)} \right| = \ln \left( \frac{2 - PR}{PR} \frac{1}{1} \right) = \ln \left( \frac{2 - PR}{PR} \right)\]