MATH 404 - Homework 1 Solutions

Write the definition of a group.

A group \( G \) consists of a set of elements and a binary operation "\( * \)" that relates two elements to a third.

The axioms for a group are:

A1. For all elements \( x \) and \( y \), \( z \in G \):

\[
(1) \quad x * y = y * x
\]

A2. The binary operation is associative: \( \forall x, y, z \in G \):

\[
(2) \quad (x * y) * z = x * (y * z)
\]

A3. There is a special element \( e \in G \) s.t. \( x * e = x \) for all \( x \in G \)

The element \( e \) is called the identity of \( G \).

A4. Given \( x \in G \), there is an element \( x^{-1} \in G \) s.t. \( x * x^{-1} = e \)

The element \( x^{-1} \) is called the inverse to \( x \).

Let \( G = \{ f: \mathbb{R} \rightarrow \mathbb{R} \mid \exists g: \mathbb{R} \rightarrow \mathbb{R} \text{ s.t. } fog = gof = \text{id}_{\mathbb{R}} \} \) be the set of invertible functions from \( \mathbb{R} \) to \( \mathbb{R} \).

Prove that \( G \) is a group under the operation \((f, g) \rightarrow fog\).

1. \( fog \) is invertible with inverse \( g^{-1} \circ f^{-1} \):

\[
(\text{Check:)} \quad (fog) \circ (g^{-1} \circ f^{-1}) = f \circ (g \circ g^{-1}) \circ f^{-1} = f \circ \text{id} \circ f^{-1} = fof^{-1} = f
\]

(Uses associativity of \( * \).)

2. \( \circ \) is associative:

\[
(\text{if } \circ \text{ is associative, then } ((fog) \circ h)(x) = (fog)(h(x)) = f(g(h(x)))
\]

\[
(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x)))
\]

3. \( \text{id}_{\mathbb{R}}: x \rightarrow x \) gives the identity element.

4. Each \( f \) has an inverse \( f^{-1} \) by definition of \( G \).
12. Exercise 1.4.6. If \( x, y, z \in G \) and \( xz = yz \) then \( x = y \).

Proof: \( xz = yz \implies (xz)z^{-1} = (yz)z^{-1} \)

But \((xz)z^{-1} = x(zz^{-1})\) by associativity,

\[ = xe \quad \text{by properties of inverse} \]

\[ = x \quad \text{by property of identity} \]

And likewise \((yz)z^{-1} = y\)

\[ \therefore \ x = y \quad \text{as claimed} \]

Exercise 1.4.7. If \( x \in G \), \( xe = eox \).

Proof: It suffices to show that \((xe)x^{-1} = (eox)x^{-1}\) by the claim.

But \((xe)x^{-1} = (x)x^{-1}\) by property of \(e\).

\[ = e \quad \text{by property of inverse} \]

And \((eox)x^{-1} = e(xox^{-1})\) by associativity.

\[ = e \cdot e \quad \text{by property of inverse} \]

\[ = e \quad \text{by property of } e \]

\[ \therefore \text{the claim holds} \]

Exercise 1.4.8. The identity element is unique.

Proof: Suppose \( e' \) an element \( e' \) st. \( xe' = x \) \( \forall x \in G \).

We want to show \( e' = e \).

\[ e = e' = e \quad \text{by definition of } e' \]

\[ e = e'e = e' \quad \text{by } \text{Ex 14.7} \]

\[ = e' \quad \text{by definition of } e \]

\[ \therefore e = e' \quad \text{as claimed} \]
Four-point geometry.

Exercise 1.5.4. Show that there are exactly six lines.

A2 = 1. For each pair of points there is a unique line containing those two points.

A3 = 2. No line has more than two points.

\[ \therefore \] there is a different line for each pair of points.

But also A3 \(\Rightarrow\) each line has a pair of points.

\[ \therefore \] there are exactly as many lines as there are pairs of points.

A4 = there are exactly 4 points and hence \(\binom{4}{2} = 6\) pairs of points.

\[ \therefore \] there are exactly six lines, as claimed.

Exercise 1.5.5. Show that each point belongs to exactly 3 lines.

By A1 above, each point belongs to exactly one line for each pair of points that it is part of (A2 and A3).

Since there are four points (A1), each point is part of exactly three pairs.

Each point is on exactly 3 lines.

Exercise 1.5.6. Show that exactly 4-point geometry is consistent relative to Euclidean geometry.

The following is a model of 4-point geometry in Euclidean geometry.

Check 1: There are exactly 4 points.

2: Between any two points there is exactly one line.

3: Each line contains exactly two points.
Exercise 1.5.7  Is a tetrahedron a model of 4 point geometry?

Check the axioms:

1. There exist 4 points.
2. For each pair of points there is exactly one line.
3. Each line contains exactly two points.

The tetrahedron is a model.