Math 402: Final Exam
Fall semester 2016

- Do not forget to write your name and netid on top of this page.
- No notes, books, calculators, or other exam aids are allowed. You may use a ruler and colored pens if you wish.
- Turn your cell phones off and put them away. No use of cell phones or other communication devices during the exam is allowed.
- Write your answers clearly and fully on the sheets provided. If you need additional paper, raise your hand.
- Do not tear pages off of this exam. Doing so will be considered cheating.
- The exam consists of 6 problems and 11 pages. Check that your exam is complete.
- You have 3 hours to complete the exam.

Good luck!!

<table>
<thead>
<tr>
<th>Problem</th>
<th>1</th>
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<tbody>
<tr>
<td>Total possible</td>
<td>32</td>
<td>12</td>
<td>24</td>
<td>20</td>
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<td>16</td>
<td>124</td>
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<td>Your points</td>
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Problem 1: (32 Points) Consider the two scaling transformations \(S_1, S_2 : \mathbb{R} \to \mathbb{R}\) given by

\[
\begin{align*}
S_1((x, y)) &= \left( \frac{1}{2}x + \frac{1}{2}y, -\frac{1}{2}x + \frac{1}{2}y \right), \\
S_2((x, y)) &= \left( -\frac{1}{2}x + \frac{1}{2}y, -\frac{1}{2}x - \frac{1}{2}y + 1 \right).
\end{align*}
\]

1. Let \(A = A_0\) be the compact set given by the line segment from \((0, 0)\) to \((0, 1)\) in \(\mathbb{R}^2\).

Recursively define \(A_n = S_1(A_{n-1}) \cup S_2(A_{n-1})\) for all integers \(n \geq 1\). Draw \(A_1 = S_1(A_0) \cup S_2(A_0)\), making sure to label the coordinates of the endpoints of the segments in your drawing.
2. Now that you’ve drawn this picture, can you decompose $S_1$ as a composition of rotations, dilations, and/or translations? First write it in words, then write down a matrix description for each of the functions in your decomposition, and finally compose them to check that you got the correct answer. In particular, what is the scaling ratio of $S_1$?

3. Now apply $S_1$ and $S_2$ to each of the segments in your drawing of $A_1$, to compute $A_2$. Draw this too, again labelling the endpoints of all segments with coordinates.
4. Compute the total length of each of the curves \( A_0, A_1, \) and \( A_2 \). What will be the length of \( A_n \)?

5. The sequence of sets \( \{A_n\} \) converges to some set \( A_\infty \). What is the total length of \( A_\infty \)? By construction, \( S_1 \) and \( S_2 \) will be self-similarities of this set. Compute its similarity dimension—you can do this using \( S_1, S_2, \) or (if you like) some composition of them such as \( S_1 \circ S_1 \). Either way, make sure you simplify your answer as much as possible.
Problem 2: (12 Points)

1. Let $F = \mathbb{Q} \cap [0,1]$. Prove that $F$ is self-similar, by writing down an appropriate scaling transformation $S$. What is the similarity dimension of $F$?

2. Let $S, T : \mathbb{R}^n \to \mathbb{R}^n$ be scaling transformations, with scaling ratios $r_S$ and $r_T$ respectively. It is not hard to show (and you can assume without proof) that the composition $S \circ T$ is again a scaling transformation, now with scaling ratio $r_S \times r_T$. In particular, $S^2$ is a scaling transformation with ratio $r_S^2$. Suppose that $S$ is a self-similarity of $F \subset \mathbb{R}^n$, and that it can be used to calculate that the similarity dimension of $F$ is equal to some number $d$. Argue that $S^2$ is again a self-similarity of $F$, and prove that the definition of similarity dimension of $F$ with respect to $S^2$ is still $d$. 


Problem 3: (24 Points) Consider the Klein disk with center $O$, and follow the instructions below.

1. Draw a diameter $d$.

2. Draw a line $\ell$ which is perpendicular to $d$ and not a diameter. Label the intersection point $A$. Let $B$ be another point on $\ell$.

3. Draw the pole $X$ of $\ell$. What do you know about $d$ and $X$?

4. Construct a line though $B$ perpendicular to $\ell$.

5. Now we want to construct a Saccheri quadrilateral with vertices $O, A, B$, and some other point $D$. How do you find $D$? (Describe your procedure in words as well as illustrating it on your picture.)

6. $O$ and $D$ are related by circle inversion with respect to some circle $c$. What is that circle? What relationship do you then have between cross-ratios involving $O$ and/or $D$?
7. Let \( r_1 \) be reflection across the line through \( A \) and \( D \), and let \( r_2 \) be reflection across the line through \( O \) and \( B \). Consider \( f = r_1 \circ r_2 \). How many fixed points does \( f \) have? Is it orientation-preserving? What is its order?
Problem 4: (20 Points)

1. For each of the following figures, can it be used to tile the hyperbolic plane? If not, explain why not.
   (a) A regular triangle with angles 60°.
   (b) A regular triangle with angles 55°.
   (c) A regular triangle with angles 45°.

2. A regular quadrilateral with angles 72° can be used to tile the hyperbolic plane. Let $ABCD$ be such a quadrilateral. Recall that in a project, you used a series of hyperbolic isometries to generate copies of figures to cover the Poincaré disk.

As a first step, we want a quadrilateral to the right of $ABCD$ sharing side $CD$. Which of the following is a better way to start, or are they all equally good? Justify your answer.
   (a) Let $m$ be the line through $CD$, and let $T_1$ be hyperbolic reflection across $m$. Apply $T_1$ to $ABCD$.
   (b) Let $T_2$ be clockwise hyperbolic rotation by angle 72° about the point $C$. Apply $T_2$ to $ABCD$.
   (c) Let $M$ be the midpoint of $ABCD$ (you can construct $M$ by taking the intersection of the diagonal lines $AC$ and $BD$). Let $T_3$ be clockwise hyperbolic rotation around $M$ by 90°. Apply $T_3$ to $ABCD$.
   (d) Let $\ell$ be the line through $BC$, and let $x$ be the hyperbolic distance from $B$ to $C$. Let $T_4$ denote hyperbolic translation along $\ell$ by distance $x$. Apply $T_4$ to $ABCD$.
   (e) Let $T_5$ be counterclockwise hyperbolic rotation by angle 60° about the point $D$. Apply $T_5$ to $ABCD$. 
3. The resulting tiling of the hyperbolic plane as in (2) is of type \((n, k)\). What is \(n\)? What is \(k\)?
**Problem 5:** (20 Points)

Are the following statements true or false? Indicate your answer neatly in the table.

*No partial credit will be given.*

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<th>True</th>
<th>False</th>
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<tbody>
<tr>
<td>(a)</td>
<td>Let $ABΩ$ be an omega-triangle and assume that the angles at $A$ and $B$ are congruent. Then the line from $Ω$ to the midpoint $M$ of the side $AB$ is <em>always</em> perpendicular to that side.</td>
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<td>(b)</td>
<td>Consider a figure in the Poincaré disk which has two ordinary vertices $A$ and $B$, and two omega-point vertices. Then the sum of the angles at $A$ and $B$ must be less than $360^\circ$.</td>
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<td>(c)</td>
<td>Let $ℓ$ be a hyperbolic line and $m$ a line perpendicular to $ℓ$. For a point $P$ on $ℓ$, let $a(P)$ denote the angle of parallelism to $ℓ$ at $P$. Suppose that $Q$ is another point on $ℓ$ such that $a(Q) &gt; a(P)$. Then $P$ must lie between $Q$ and $ℓ$.</td>
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<td>(d)</td>
<td>Let $ABC$ be an equilateral triangle in the hyperbolic disk. Then there is a unique Möbius transformation $f$ which sends $A$ to $A$, $B$ to $C$, and $C$ to $B$.</td>
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<td>(e)</td>
<td>The set of Möbius transformations which map the unit disk to the upper half-plane forms a group.</td>
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<td>(f)</td>
<td>Let $ABΩ$ be an omega-triangle. Let $ℓ$ be the line through $AΩ$ and let $m$ be the line through $BΩ$. Let $r_ℓ$ and $r_m$ denote reflections across these lines. Then $r_ℓ \circ r_m$ has infinite order.</td>
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<td>(g)</td>
<td>Let $c$ be a hyperbolic circle and $f \neq \text{id}$ a hyperbolic isometry such that $f(c) = c$. Then $f$ is either a rotation or a reflection.</td>
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<td>(h)</td>
<td>Stereographic projection is an example of a Möbius transformation.</td>
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<td>(i)</td>
<td>To express any isometry $f$ of the Poincaré disk, it is enough to compose Möbius transformations with reflections across the line $x = y$.</td>
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<td>(j)</td>
<td>Let $ABCD$ be a Lambert quadrilateral with an acute angle at the vertex $D$. Let $BD$ be the diagonal. Then the triangles $ABD$ and $CBD$ are sometimes but not always congruent.</td>
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**Problem 6:** (16 Points)

1. Write the definition of distance in the Poincaré disk.

2. Let \( \ell \) be a line and \( P \) a point. Write down a definition for the reflection of \( P \) across \( \ell \)—your definition should be valid in Euclidean and hyperbolic geometry.

3. Let \( c \) be a circle and \( \ell \) a line. We know that there exists a Möbius transformation \( f \) which takes \( c \) to \( \ell \). Is \( f \) unique? Justify your answer.

4. What are the symmetries of the following frieze patterns?
   (a.) \[ \ldots \, \hat{\cdot} \, \hat{\cdot} \, \hat{\cdot} \, \hat{\cdot} \, \hat{\cdot} \, \hat{\cdot} \, \ldots \]

   (b.) \[ \ldots \, \hat{\cdot} \, \hat{\cdot} \, \hat{\cdot} \, \hat{\cdot} \, \hat{\cdot} \, \hat{\cdot} \, \ldots \]