Thursday, September 6  **  Cross product and applications

1. *On your whiteboard, draw two vectors \( \mathbf{a} \) and \( \mathbf{b} \) with their tails touching and with an acute angle, \( \theta \), between them. If we take the cross product, \( \mathbf{a} \times \mathbf{b} \), we get another vector which is perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \). To figure out which direction \( \mathbf{a} \times \mathbf{b} \) points, we do the following.

   - Take your right hand and point your fingers in the direction of the vector \( \mathbf{a} \).
   - Bend your fingers in the direction of the vector \( \mathbf{b} \) (you may have to turn your hand over).
   - The direction your thumb is now pointing is the direction that \( \mathbf{a} \times \mathbf{b} \) points. This is called the right-hand rule.

   (a) Which direction does \( \mathbf{a} \times \mathbf{b} \) point for the vectors you drew?

   (b) Switch the labels of the vectors you drew. Now which direction does \( \mathbf{a} \times \mathbf{b} \) point?

2. *Again on your whiteboard, draw two vectors \( \mathbf{a} \) and \( \mathbf{b} \) with their tails touching and with an acute angle, \( \theta \), between them. Draw the parallelogram which is spanned by these two vectors.

   (a) Find a formula for the area of the parallelogram spanned by \( \mathbf{a} \) and \( \mathbf{b} \) which includes \( \theta \).

   (b) The length of the vector \( \mathbf{a} \times \mathbf{b} \) is exactly equal to the area of the parallelogram spanned by \( \mathbf{a} \) and \( \mathbf{b} \). Use your formula from part (a) to write an equation involving the cross product which is similar to the equation \( \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos \theta \) which we had for the dot product.

   (c) How are these equations similar and how do they differ?

3. *Given two vectors \( \mathbf{a} \) and \( \mathbf{b} \), use parts 1. and 2. to describe the vector \( \mathbf{a} \times \mathbf{b} \). (What two pieces of information do we need to describe a vector?)

4. *Let \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) be the standard basis vectors. On your whiteboard, draw a large \( xy \)-plane and label the positive \( x \) and \( y \)-axes. On this plane, draw \( \mathbf{i} \) and \( \mathbf{j} \) so that they have the same length as a large whiteboard marker. Now, stand a large whiteboard marker up at the origin to represent \( \mathbf{k} \). Using only the properties of the cross product described in 1. and 2., determine:

   (a) \( \mathbf{i} \times \mathbf{j} = \)  
   (b) \( \mathbf{i} \times \mathbf{k} = \)  
   (c) \( \mathbf{j} \times \mathbf{k} = \)  
   (d) \( \mathbf{j} \times \mathbf{i} = \)  
   (e) \( \mathbf{k} \times \mathbf{i} = \)  
   (f) \( \mathbf{k} \times \mathbf{j} = \)

(Continued on back!)
We saw in lecture that if \( \mathbf{u} = (u_1, u_2) \) and \( \mathbf{v} = (v_1, v_2) \), then the \( 2 \times 2 \) determinant is defined as
\[
\begin{vmatrix}
u_1 & u_2 \\
v_1 & v_2 \\
\end{vmatrix} = u_1v_2 - u_2v_1
\]
and calculates the signed area of the parallelogram spanned by \( \mathbf{u} \) and \( \mathbf{v} \). The \( 3 \times 3 \) determinant is defined using the \( 2 \times 2 \) determinant as follows:
\[
\begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3 \\
\end{vmatrix} = a_1 \begin{vmatrix}
b_2 & b_3 \\
c_2 & c_3 \\
\end{vmatrix} - a_2 \begin{vmatrix}
b_1 & b_3 \\
c_1 & c_3 \\
\end{vmatrix} + a_3 \begin{vmatrix}
b_1 & b_2 \\
c_1 & c_2 \\
\end{vmatrix}.
\]
So, if \( \mathbf{a} = (a_1, a_2, a_3) \) and \( \mathbf{b} = (b_1, b_2, b_3) \), then the cross product \( \mathbf{a} \times \mathbf{b} \) is calculated as follows:
\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix}
i & j & k \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
\end{vmatrix} = i \begin{vmatrix}
a_2 & a_3 \\
b_2 & b_3 \\
\end{vmatrix} - j \begin{vmatrix}
a_1 & a_3 \\
b_1 & b_3 \\
\end{vmatrix} + k \begin{vmatrix}
a_1 & a_2 \\
b_1 & b_2 \\
\end{vmatrix}.
\]

5. * Suppose \( \mathbf{a} = (3, -2, 0) \) and \( \mathbf{b} = (4, 1, 2) \).
   (a) Compute \( \mathbf{a} \times \mathbf{b} \).
   (b) Compute \( \mathbf{b} \times \mathbf{a} \).
   (c) How do your answers in (a) and (b) compare? Use your (right) hand to help explain this.

6. *If two vectors are parallel, what can we say about their cross product? Does this statement work in reverse? How does this statement compare and contrast to the analogous statement about orthogonal vectors and the dot product?

7. *Find an equation for the plane \( P \) which contains the points \( A(1, 4, 3) \), \( B(-2, 3, 5) \), and \( C(0, -6, 1) \).

8. *Using only problem 4. and the properties of the cross product discussed in lecture, find
\[
(i + k) \times (k - 2j).
\]