Parametric curves defined using vector arithmetic.

Note: Problems marked with an asterisk (*) are Merit problems. All other problems come from the regular Math 241 worksheet, although I may have modified them slightly. You will find the original discussion worksheets and their solutions on the Math 241 course webpage.

1. Let \( f(x) = x^2 + x - 2 \).
   
   (a) Graph the equation \( y = f(x) \). (By hand, then check with a calculator if you want.)
   
   (b) Find the slope and equation of the tangent line to \( y = f(x) \) when \( x = 2 \). Draw the tangent line on your picture.
   
   (c) Draw a vector in \( \mathbb{R}^2 \) that describes the direction of the line. Find a numeric representation of your vector.

2. Consider the curve given parametrically by
   \[
   \begin{align*}
   x(t) &= t \\
   y(t) &= t^2 + t - 2
   \end{align*}
   \]
   for \( 0 \leq t < 4 \).
   
   (a) Sketch the curve. How does this graph differ from your graph in Problem 1(a)?
   
   (b) Consider the vectors formed by the pair \( (x(t), y(t)) \). Anchoring the vectors at the origin, sketch on your graph the vectors at time \( t = 0,1,2,3 \).
   
   (c) Now consider the vectors formed by \( (x'(t), y'(t)) \). Evaluate \( (x'(t), y'(t)) \) at time \( t = 2 \), what does the vector represent? Hint: Graph it on the curve at the point \( (x(2), y(2)) \).
   
   (d) Imagine that the curve is the path of a moving particle. What is the speed of the particle when \( t = 2 \)?

3. *What are the similarities and differences between the vector \( \mathbf{v} = (2, -3, 0) \) and the point \( (2, -3, 0) \)?

4. (a) *Draw the rectangular prism which has one vertex at the origin and an opposite vertex at the point \( (4,3,1) \).
   
   (b) *Use part (a) to draw the vector \( (4, 3, 1) \) emanating from the origin.
   
   (c) *Use the "head-to-tail" geometric addition method and your picture to interpret the equation: \( (4,3,1) = (0,3,0) + (0,0,1) + (4,0,0) \).
   
   (d) *On the same picture and in a different color, repeat parts (a) and (b) for \( (1, 1, -2) \).
   
   (e) *On the same picture, draw the line in \( \mathbb{R}^3 \) that goes through the points \( (4,3,1) \) and \( (1, 1, -2) \).

5. (a) Sketch the vector emanating from the origin ending at the point \( (-5, 2) \) in \( \mathbb{R}^2 \).
   
   (b) On the same graph and using the “head-to-tail” geometric addition method, draw the vector \( (-5, 2) + (3, -1) \).
   
   (c) Do the same for \( (-5, 2) + 2(3, -1) \).
(d) Do the same for $(-5, 2) + (-1)(3, -1)$.

(e) If we allow the scalar multiplying the vector $(3, -1)$ to vary, what geometric object is described by the parametric equation $(-5, 2) + t(3, -1)$ for all $t$?

6. Consider the set of points in $\mathbb{R}^3$ defined by the parametric equation

$$I(t) = (-5 + 2t, 2 + 3t, 1 - t) \text{ for all } t.$$ 

(a) Using the properties of vector arithmetic, factor $I(t)$ into the form $p + tv$ where $p$ and $v$ are vectors in $\mathbb{R}^3$.

(b) Using the factored form (and your technique from Problem 5) sketch this object in $\mathbb{R}^3$. Geometrically, what does this parametric equation describe?

(c) Why is the vector $v$ in your factored form referred to as the direction vector?

7. *Two (nonzero) vectors are said to be orthogonal or perpendicular if they form an angle of $\frac{\pi}{2}$. If two vectors are orthogonal, what can we say about their dot product? Does your statement always work in reverse? If not, in what situations does it work in reverse?

8. Let $a = (-\sqrt{3}, 0, -1, 0)$ and $b = (1, 1, 0, 1)$ be vectors in $\mathbb{R}^4$.

(a) Find the distance between the points $(-\sqrt{3}, 0, -1, 0)$ and $(1, 1, 0, 1)$.

(b) Find the length of the vector $a - b$. How does this compare to part (a)?

(c) Find the angle between $a$ and $b$.

9. *Let $a = (2, 1)$ be a vector in $\mathbb{R}^2$.

(a) Draw the vector $a$ emanating from the origin. What unit vector $b$ in $\mathbb{R}^2$ will maximize $a \cdot b$? Draw this vector emanating from the origin.

(b) What unit vector $b$ in $\mathbb{R}^2$ will minimize $a \cdot b$? Draw this vector emanating from the origin.

(c) What unit vector $b$ will minimize $|a \cdot b|$? Draw this vector emanating from the origin. Is this the only such vector?

(d) Now, let $b = (2, 1, 0)$ be a vector in $\mathbb{R}^3$. Use part (c) to find two unit vectors in $\mathbb{R}^3$ perpendicular to $b$. Are these the only unit vectors in $\mathbb{R}^3$ perpendicular to $b$? What geometric object is described by the set of all unit vectors in $\mathbb{R}^3$ emanating from the origin which are perpendicular to $b$? Draw a picture to help you.

(e) What geometric object is described by the set of all vectors (of any length) in $\mathbb{R}^3$ emanating from the origin which are perpendicular to $b$?

10. *Let $a = (3, 4, 5)$ be the vector in $\mathbb{R}^3$ anchored at the origin. What geometric object is described by the set of vectors $v$ in $\mathbb{R}^3$ anchored at the origin for which $|v - a| = 4$?