Small weights in divisible codes.
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Abstract: We give a new description of the relation on the low weight coefficients of self-dual divisible codes. We use them to give a short and elementary proof of bounds for self-dual codes that were obtained by Krasikov-Litsyn and by Rains.

Let $A(x, y)$ be a self-dual weight enumerator. We formulate the relation among the low weight coefficients as follows

$$[y^d(x-y)^d]A(x,y)((x-y)^2 - qy^2) = 0$$

That is to say, if we write the right hand side in variables $x - y$ and y then the coefficient at $y^d(x-y)^d$, for $2d = n + 2$, vanishes. For $u = x - y, v = y$,

$$[u^d v^d]A(u+v, v)(u^2 - qv^2) = 0$$

Theorem:

(Type III/IV)

Let $A(x, y) = F(x^c - y^c, y^c)$ be a self-dual divisible weight enumerator of length n. Define N and D such that $(c + 1)D = n + c(c + 1) = N$. Let $u^c = x^c - y^c, v^c = y^c$.

$$\begin{aligned} \textit{Type III}: \quad & [v^D u^{3D}]F(u^3, v^3)H^*(u^3, v^3) = 0, \\ & \text{for } H^*(u^3, v^3) = (u^6 - 18u^3v^3 + 27v^6)(u^3 + 9v^3)^2. \\ \textit{Type IV}: \quad & [v^D u^{2D}]F(u^2, v^2)H^*(u^2, v^2) = 0, \\ & \text{for } H^*(u^2, v^2) = (u^2 - 8v^2)(u^2 + 4v^2)^2. \end{aligned}$$

(Type I/II)

Let $A(x, y) = F((x^c - y^c)^2, x^c y^c)$ be a symmetric self-dual divisible weight enumerator of length n. Define N and D such that $(c+2)D = n+c(c+2) = N$.

Let $u^c = (x^c - y^c), v^{2c} = x^c y^c$.

$$
\begin{aligned}
Type\ I: \quad & [v^{2D}u^{2D}]A(u^4, v^4)H^*(u^4, v^4) = 0, \\
& \text{for } H^*(u^4, v^4) = (u^4 - 4v^4)(u^4 + 4v^4). \\
Type\ II: \quad & [v^{2D}u^{4D}]A(u^4, v^4)H^*(u^4, v^4) = 0, \\
& \text{for } H^*(u^4, v^4) = (u^8 - 32v^8)(u^8 + 16v^8)^2.
\end{aligned}
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Comment: The proof reduces to a simple verification when using invariant theory. We also have a direct proof that avoids invariant theory. The relations described by the theorem are precisely the linear programming relations used by Krasikov-Litsyn, and by Rains, to obtain their asymptotic bounds. Those bounds follow as a corollary to the theorem.