Distance bounds for AG codes

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September 28, 2009
Workshop on Sequences, Codes and Curves
Antalya, Turkey

Joint work with
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Selection of references

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1 – The role of the minimum distance in coding theory, secret sharing, and secure computation

2 – Some geometry

Delta sets

3 – Basic methods to obtain bounds

AB method / ABZ Theorem

Shifting method / Main Theorem

4 – Comparison and numerical results

Coding theory

Let $C \subseteq \mathbb{F}^n$ be a linear code. The minimum distance d of the code is the minimum of $d(x, y) = \#\{i : x_i \neq y_i\}$ for any two distinct vectors $x, y \in C$.

Given a vector $c \in C$ and a subset $A \subset \{1, 2, \ldots, n\}$, the values of c on A give the following information about values of c outside A (as a function of $|A|$):

$$\underbrace{0 \ \cdots\cdots \ d^{\perp} - 2}_{\text{No information}} \ \underbrace{\cdots\cdots}_{?} \ \underbrace{n - d + 1 \ \cdots\cdots \ n}_{\text{Full information}}$$

Secret sharing (Quotient code version [D-Park'08])

Let $C_0 \subseteq C$ be a subcode with $\dim C/C_0 = 1$ and let $x \in C \backslash C_0$. For given $c \in C$, we seek to determine $s \in \mathbb{F}$ such that $c - s \cdot x \in C_0$.

Given a vector $c \in C$, the values of c on A give the following information about s (as a function of $|A|$):

$$\underbrace{0 \ \cdots\cdots \ d_0^{\perp} - 1}_{\text{No information about } s} \qquad \underbrace{\cdots\cdots}_{?} \qquad \underbrace{n - d_0 + 1 \ \cdots\cdots \ n}_{\text{Full information about } s}$$

Here $d_0 = d(C/C_0)$ and $d_0^{\perp} = d(C_0^{\perp}/C^{\perp})$.

Relation between $d = d(C)$ and $d_0 = d(C/C_0)$.

$$\begin{aligned} d(C/C_0) &= \min\{\, d(x + C_0, y + C_0) \;\; : \\ &\qquad\qquad \text{distinct } x + C_0, y + C_0 \in C/C_0 \,\}. \\ &= \min\{\, \mathrm{wt}(c) : c \in C \backslash C_0 \,\}. \end{aligned}$$

So that

$$d(C) = \min\{\, d(C/C_0), d(C_0) \,\}$$

Let the code C be defined by evaluation of $1, x, x^3, x^4, x^7$ in the ninth roots of unity.

$$C = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & \alpha & \cdots & \cdots & \alpha^7 & \alpha^8 \\ 1 & \alpha^3 & \cdots & \cdots & \alpha^3 & \alpha^6 \\ 1 & \alpha^4 & \cdots & \cdots & \alpha & \alpha^5 \\ 1 & \alpha^7 & \cdots & \cdots & \alpha^4 & \alpha^2 \end{pmatrix}$$

And let

$$C_0 = \mathsf{Ev}(x, x^3, x^4, x^7) \subset C = \mathsf{Ev}(1, x, x^3, x^4, x^7).$$

Then

$$d(C) = 3, \quad d(C/C_0) = 4, \quad d(C_0) = 3.$$

For codes A and B of same length n, let

$$A * B = \langle (a_i b_i) : (a_i) \in A, (b_i) \in B \rangle.$$

A quotient C/C_0 of dimension one is said to be multiplicative if $C*C/C_0*C$ is of dimension one. This is the case if and only if $x * x \in C * C \backslash C_0 * C$, for $x \in C \backslash C_0$.

Secure multiplication of the product $s \cdot s'$ from the factors s and s' requires that

- $d_0^{\perp}(C) > t$ (privacy: no information about the factors s or s' when $|A| \leq t$))

- $d_0(C * C) > t$ (security: full information about the product $s \cdot s'$ when $|A| \geq n - t$)

Geometric Goppa codes

For n distinct rational points $P_1, \ldots, P_n$ on a curve X, and for disjoint divisors $D = P_1 + \cdots + P_n$ and G, the geometric Goppa codes $C_L(G, D)$ and $C_\Omega(G, D)$ are defined as the images of the maps

$$\begin{aligned} \alpha_L \ : \ L(G) &\longrightarrow \mathbb{F}^n \\ f &\mapsto (\, f(P_1), \ldots, f(P_n)\,) \end{aligned}$$

$$\begin{aligned} \alpha_\Omega \ : \ \Omega(G - D) &\longrightarrow \mathbb{F}^n \\ \omega &\mapsto (\, \mathrm{Res}_{P_1}(\omega), \ldots, \mathrm{Res}_{P_n}(\omega)\,). \end{aligned}$$

Goppa bound

$$d(C_L(D,G)) \;=\; \min\{\deg A : 0 \le A \le D \;\wedge\; L(G-D+A) \neq L(G-D)\}.$$

$$d(C_\Omega(D,G)) \;=\; \min\{\deg A : 0 \le A \le D \;\wedge\; L(K-G+A) \neq L(K-G)\}.$$

$$d(C_L(D,G)) \;\geq\; \min\{0, \deg(D-G)\}$$

$$d(C_\Omega(D,G)) \;\geq\; \min\{0, \deg(G-K)\}$$

For a rational point P, for D the sum of the $N-1$ rational points outside P, and for G of degree $2g+t$,

$$d_0^{\perp}(C) \geq d^{\perp}(C_0) \geq t+1,$$
$$d_0(C*C) \geq d(C*C) \geq (N-1)-(4g+2t).$$

[Chen-Cramer'06] For a curve over $\mathbb{F}_q$ of genus g with N rational points, secure multiplication using codes of length $n = N-1$ is possible for

$$n > 3t+4g, \quad \text{or} \quad t/n < (1-4g/n)/3.$$

For a projective line, a Hermitian curve, or an optimal asymptotic tower over $\mathbb{F}_q$, $q = r^2$,

$t/n < 1/3$	$n = q$
$t/n < (1-2/(r+1))/3$	$n = r^3$
$t/n \lesssim (1-4/(r-1))/3$	$n \to \infty$.

2 – Some Geometry : Delta sets.

In the theory of linear secret sharing schemes, the subsets of $\{1, 2, \ldots, n\}$ are partitioned into qualified and unqualified subsets.

$$\Gamma_0(C) = \{A : A \text{ has acces to } s\}$$
$$\Delta_0(C) = \{A : A \text{ has no access to } s\}$$

In particular,

$$\{A : |A| \geq n - d_0 + 1\} \subseteq \Gamma_0(C)\}$$
$$\{A : |A| \leq d_0^{\perp} - 1\} \subseteq \Delta_0(C)\}$$

Delta sets in Gröbner basis theory.

Let $\mathbb{F}[x, y]$ be the coordinate ring of the affine plane, with ordered basis

$$1 < x < y < x^2 < xy < y^2 < x^3 < \cdots$$

For a set of points $\{P_1, P_2, \ldots, P_n\}$ with vanishing ideal

$$I = \{f : f(P_i) = 0, \text{ for } i = 1, 2, \ldots, n\},$$

the Delta set is a greedy basis for $\mathbb{F}[x, y]/I$ with respect to the given ordering. In general, $|\Delta| = n$.

Delta sets for divisors on curves.

Let $X : y^4 + y = x^5$ be the Hermitian curve of genus $g = 6$ over the field $\mathbb{F} = \mathbb{F}_{16}$. The functions x and y have poles only at P, of order 4 and 5, respectively. They generate the affine ring $\mathbb{F}[x, y] = \mathcal{O}$ of all functions with poles only at P. The ordered basis

$$1 < x < y < x^2 < xy < y^2 < x^3 < x^2y < \cdots$$

corresponds to an ordering

$$0 < 4P < 5P < 8P < 9P < 10P < 12P < 13P < \ldots$$

of the Weierstrass nongaps at P.

For divisors $0 \leq C \leq D$ such that C, D disjoint from P,

$$\Delta(D) \supseteq \Delta(C).$$

For the curve $X : y^4 + y = x^5$, let $Q = (0,0), R = (0,1)$.

$$\begin{array}{lll} \Delta(Q+R) & = \{1, y\}, & \text{or } \{0, 5\}. \\ \Delta(2Q+2R) & = \{1, x, y, xy\}, & \text{or } \{0, 4, 5, 9\}. \\ \Delta(3Q+3R) & = \{1, x, y, x^2, xy, x^2y\}, & \text{or } \{0, 4, 5, 8, 9, 13\}. \end{array}$$

Let $I(C) \subseteq \mathcal{O}$ be the vanishing ideal of C, for $C \geq 0$ disjoint from P. A basis for $\mathcal{O}/I(C)$ is given by $\{f_i : i \in \Delta(C)\}$, where

$$i \in \Delta(C) \;\Leftrightarrow\; \begin{cases} (1)\ L(iP) \neq L((i-1)P), \text{ and} \\ (2)\ L(iP - C) = L((i-1)P - C). \end{cases}$$

For $C \geq 0$ not disjoint from P, there are no functions in $\mathcal{O}$ vanishing on C. We can however compute $\Delta(C)$, as well as the set $\Delta'(C)$, where

$$i \in \Delta'(C) \;\Leftrightarrow\; \begin{cases} (1')\ L(iP) = L((i-1)P), \text{ and} \\ (2')\ L(iP - C) \neq L((i-1)P - C). \end{cases}$$

In general, $\deg C = |\Delta| - |\Delta'|$.

For $C = 2P + Q$,

$$\Delta(2P+Q) = \{0,4,5,8,9,13\},$$
$$\Delta'(2P+Q) = \{6,7,11\}.$$

	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
(1)		$*$				$*$	$*$			$*$	$*$	$*$		$*$	$*$
(2)								$*$	$*$			$*$	$*$	$*$	
(3)	0	1	1	0	0	1	2	1	0	1	2	2	1	0	2

(1) $L(iP) \neq L((i-1)P)$,
(2) $L(iP - C) \neq L((i-1)P - C)$,
(3) $\dim L(iP)/L(iP - C)$.

Bounds for the representation of a fractional O-ideal as quotient of integral ideals.

For any divisor C, and for divisors $D, E \geq 0$ such that $C \sim D - E$, D, E disjoint from P,

$$\deg D \geq |\Delta(C)|, \quad \deg E \geq |\Delta'(C)|.$$

Moreover,

$$\Delta(D) \supseteq \Delta(C), \quad \Delta(E) \supseteq \Delta(-C).$$

For $K = (2g-2)P$,

$$\Delta(-C) = \{K + P - iP : iP \in \Delta'(C)\}.$$

For $D, E \geq 0$ disjoint from P, such that $2P + Q \sim D - E$,

$$\Delta(D) \supseteq \Delta(2P + Q) = \{0, 4, 5, 8, 9, 13\}.$$
$$\Delta(E) \supseteq \Delta(-2P - Q) = \{0, 4, 5\}.$$

The bounds are sharp.

$$2P + Q \sim (3S + 3T) - (Q + 2R).$$

Obtained from the relations

$$5S \sim 5T \sim P + Q + R + S + T.$$

3 – Basic methods to obtain bounds for the minimum distance

AB Method (gives bounds for $d(C)$)

For fiven code C, let A and B be codes of the same length such that

$$A * B \perp C$$

Then, for all nonzero $c \in C$,

$$\mathsf{wt}(c) \geq \mathsf{rank}(c * A) + \mathsf{rank}(c * B).$$

Applying the AB Method..

Let M be a MDS code of type $[n = 5, k = 2]$ with $M \perp M * M$, say

$$M = \begin{pmatrix} 1 & \alpha & \alpha^2 & \alpha^3\alpha^4 \\ 1 & \alpha^4 & \alpha^3 & \alpha^2\alpha^1 \end{pmatrix}$$

Let

$$A = B = C = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$$

So that $C \perp A * B$. On any seven positions, A and B are of full rank 4. Thus C can not have a word of weight 7 (it is easy to see that the weights of C are $4, 5, 8, 9, 10$).

In general the AB method excludes weights that can occur in C depending on the weight hierarchies of the codes A and B.

The Hartmann-Tzeng and Roos bounds for cyclic codes appear as special cases and use only information about the minimum distances of A and B and their duals.

Let $d_i(A^\perp)$ be the $i-$th weight of $A^\perp$ and let $d_j(B^\perp)$ be the j-th weight of $B^\perp$. Then

$$\text{rank}(c * A) \geq \min\{\text{wt}(c) - (i-1), d_i(A^\perp) - i\}.$$
$$\text{rank}(c * B) \geq \min\{\text{wt}(c) - (j-1), d_j(B^\perp) - j\}.$$

If i and j are such that

$$(i-1) < d_j(B^\perp) - j \quad \text{and} \quad (j-1) < d_i(A^\perp) - i$$

then

$$\text{wt}(c) \leq (i-1)+(j-1) \text{ or } \text{wt}(c) \geq d_i(A^\perp)-i+d_j(B^\perp)-j.$$

ABZ Theorem (D-Park '08)

Let $K + C \sim A + B + Z$, for $Z \geq 0$.

If $C \sim D - E$, for $D, E \geq 0$

such that $D \cap Z = \emptyset$ then

$$\deg D \geq l(A) - l(A - C) + l(B) - l(B - C)$$

$$(= \deg C + \deg Z + l(A) - l(A + Z) + l(B) - l(B + Z))$$

Proof.

We may assume that $D \cap E = \emptyset$. With $E, Z \geq 0$ and $D \cap E = D \cap Z = \emptyset$, the natural maps

$$L(A)/L(A-D) \longrightarrow L(A+E)/L(A+E-D)$$
$$L(B)/L(B-D) \longrightarrow L(B+Z)/L(B+Z-D)$$

are well defined and injective. Therefore

$$\begin{aligned}
&\deg D \ = \\
&= \ l(A+E) - l(A+E-D) + i(A+E-D) - i(A+E) \\
&= \ l(A+E) - l(A+E-D) + l(B+Z) - l(B+Z-D) \\
&\geq \ l(A) - l(A-D) + l(B) - l(B-D) \\
&\geq \ l(A) - l(A-C) + l(B) - l(B-C).
\end{aligned}$$

For the Hermitian curve of degree 5, with $K = 10P$, let $C = 2P + Q$.

For $A = 5P, B = 5P, Z = 2P + Q$, $K + C = A + B + Z$.

Thus, if $C \sim D - E$ for divisors $D, E \geq 0$ with D disjoint from Z then D has degree at least

$$\begin{aligned}\deg D &\geq l(A) - l(A - C) + l(B) - l(B - C) \\ &= l(5P) - l(3P - Q) + l(5P) - l(3P - Q) \\ &= 3 - 0 + 3 - 0 = 6.\end{aligned}$$

The bound is sharp, $3S + 3T \sim (2P + Q) + (Q + 2R)$.

Shift bound (gives bounds for $d_0(C)$)

Let C/C_0 be an extension of $\mathbb{F}$-linear codes with corresponding extension of dual codes D/D_0 such that $\dim C/C_0 = \dim D/D_0 = 1$.

If there exist vectors $a_1, \ldots, a_w$ and $b_1, \ldots, b_w$ such that

$$\begin{cases} a_i * b_j \in D_0 & \text{for } i+j \leq w, \\ a_i * b_j \in D \backslash D_0 & \text{for } i+j = w+1, \end{cases}$$

then $d(C/C_0) \geq w$.

Proof: For a vector $c \in C \backslash C_0$, the vectors $a_1 * c, \ldots, a_w * c$ are linearly independent (with the functionals $b_1, \ldots, b_w$ as witnesses). But then c has weight $\geq w$.

For

$$C_0 = \mathsf{Ev}(x, x^3, x^4, x^7) \ \subset \ C = \mathsf{Ev}(1, x, x^3, x^4, x^7).$$

we can choose

$$\begin{aligned} a_1 = b_1 &= \mathsf{Ev}(x^5), \\ a_2 = b_2 &= \mathsf{Ev}(x^2), \\ a_3 = b_3 &= \mathsf{Ev}(x^7), \\ a_4 = b_4 &= \mathsf{Ev}(x^4). \end{aligned}$$

And $d_0(C) = d(C/C_0) \geq 4$.

Two-variable example

Who can recover $f(0,0)$?

- For $f(x,y) \in \langle 1, x, y, xy, x^2y, y^2, xy^2, x^2y^2 \rangle$.
- For $f(x,y) \in \langle 1, y, y^2, x, xy, xy^2, xy^3, xy^4 \rangle$.

$$f(x,y) \in \langle (1,x), \quad (1,x,x^2)y, \quad (1,x,x^2)y^2 \rangle$$

	b_1	b_2	b_3	b_4	b_5
a_1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
a_2	$\cdot$	$*$	$*$	$*$	$\cdot$
a_3	$\cdot$	$\cdot$	$*$	$*$	$*$

$$f(x,y) \in \langle (1,y,y^2), \quad (1,y,y^2,y^3,y^4)x \rangle.$$

	b_1	b_2	b_3	b_4	b_5
a_1	$\cdot$	$\cdot$	$*$	$\cdot$	$*$
a_2	$\cdot$	$*$	$*$	$\cdot$	$*$
a_3	$\cdot$	$*$	$\cdot$	$\cdot$	$\cdot$

$C_0 = \mathsf{Ev}(1, x, x^2, y, xy, x^2y, y^2) \ \subset \ C = \langle C_0, \mathsf{Ev}(xy^2) \rangle.$

1	x	y	xy	y^2	xy^2
x	x^2	xy	x^2y	xy^2	
y	xy	y^2	xy^2		
xy	x^2y	xy^2			
y^2	xy^2				
xy^2					

$C_0 = \mathsf{Ev}(1, x, x^2, x^3, x^4, y, xy) \ \subset \ C = \langle C_0, \mathsf{Ev}(x^2y) \rangle.$

1	x	x^2	y	xy	x^2y
x	x^2	1	xy	x^2y	
x^2	x^3	x^4	x^2y		
y	xy	x^2y			
xy	x^2y				
x^2y					

We use the shifting method to show that wt$(c) \geq 6$ for $c \in C_\Omega(K + 2P + Q, D) \backslash C_\Omega(K + 3P + Q, D)$.

We need functions $f_1, \ldots, f_6$ and $g_1, \ldots, g_6$ such that

$$f_i g_j \in L(12P + Q), \quad \text{for } i + j \leq 6.$$
$$f_i g_j \in L(13P + Q) \backslash L(12P + Q), \quad \text{for } i + j = 7.$$

Choose

$$\begin{aligned}
f_1 = g_1 &= 1 \\
f_2 = g_2 &\in L(4P) \backslash L(3P) \\
f_3 = g_3 &\in L(5P) \backslash L(4P) \\
f_4 = g_4 &\in L(8P) \backslash L(7P) \\
f_5 = g_5 &\in L(9P) \backslash L(8P) \\
f_6 = g_6 &\in L(13P) \backslash L(12P)
\end{aligned}$$

The functions $f_1, \ldots, f_6$ and their pole divisors correspond to the Delta set

$$\Delta(2P + Q) = \{0, 4, 5, 8, 9, 13\}$$

Indeed, for a pair f, g with $fg \in L(K+C+P)\backslash L(K+C)$, say

$$f \in L(A)\backslash L(A-P), g \in L(K+C+P-A) \neq L(K+C-A)$$

the divisor A meets the conditions

(1) $L(A) \neq L(A - P)$.
(2) $L(A - C) = L(A - C - P)$.

$$d(C_\Omega(D,G)) \;=$$

$$= \min\{\deg A : 0 \le A \le D \;\wedge\; \Omega(G-A) \neq \Omega(G)\}.$$

Let S and S' be finite sets of primes. For $D \cap S = \emptyset$,

$$0 \le A \le D \;\Rightarrow\; (\forall P \in S)\; L(A) \neq L(A-P).$$

For $c \in C_\Omega(D,G) \setminus \cup_{P \in S'} C_L(D, G+P')$, the support A of c satisfies

$$(\forall P \in S')\; \Omega(G-A) \neq \Omega(G+P'-A)$$

Or, for $G = K + C$,

$$(\forall P \in S')\; L(A-C) \neq L(A-C-P).$$

For a divisor C and for subsets S and S' of rational points, define

$$\begin{aligned}\gamma(C; S, S') \;=\; & \min\{\ \deg A : \\ & (\forall P \in S)\ L(A) \neq L(A - P) \\ & (\forall P \in S')\ L(A - C) \neq L(A - C - P)\ \}.\end{aligned}$$

Theorem 1 (DKP'09)

Let S, S' be finite sets of points and let Λ be the semigroup of effective divisors with support in S'. For $D \cap S = \emptyset$,

$d(C_L(D,G)) \geq$

$$\min\{\gamma(D - G + \lambda; S, S') : \lambda \in \Lambda\}\backslash\{0\}.$$

$d(C_\Omega(D,G)) \geq$

$$\min\{\gamma(G - K + \lambda; S, S') : \lambda \in \Lambda\}\backslash\{0\}.$$

Theorem 2 (DKP'09)

For a given divisor $C \sim D-E$, and for divisors $A_0, \ldots, A_n$, such that $A_i = A_{i-1} + P_i$, define the subsets

$$\Delta = \{i : L(A_i) \neq L(A_i - P) \wedge L(A_i - C) = L(A_i - C - P)\}$$

$$\Delta' = \{i : L(A_i) = L(A_i - P) \wedge L(A_i - C) \neq L(A_i - C - P)\}$$

$$S = \{i : L(D) \neq L(D - P_i)\}$$

$$S' = \{i : L(E) \neq L(E - P_i)\}.$$

Then $\deg D \geq |\Delta \cap S'| + |\Delta' \cap S| - |\Delta'|$.

In particular, $\gamma(C; S, S') \geq |\Delta|$ for

$\Delta \subseteq S'$ and $\Delta' \subseteq S$.

Special cases of Theorem 2

(1) Beelen'07

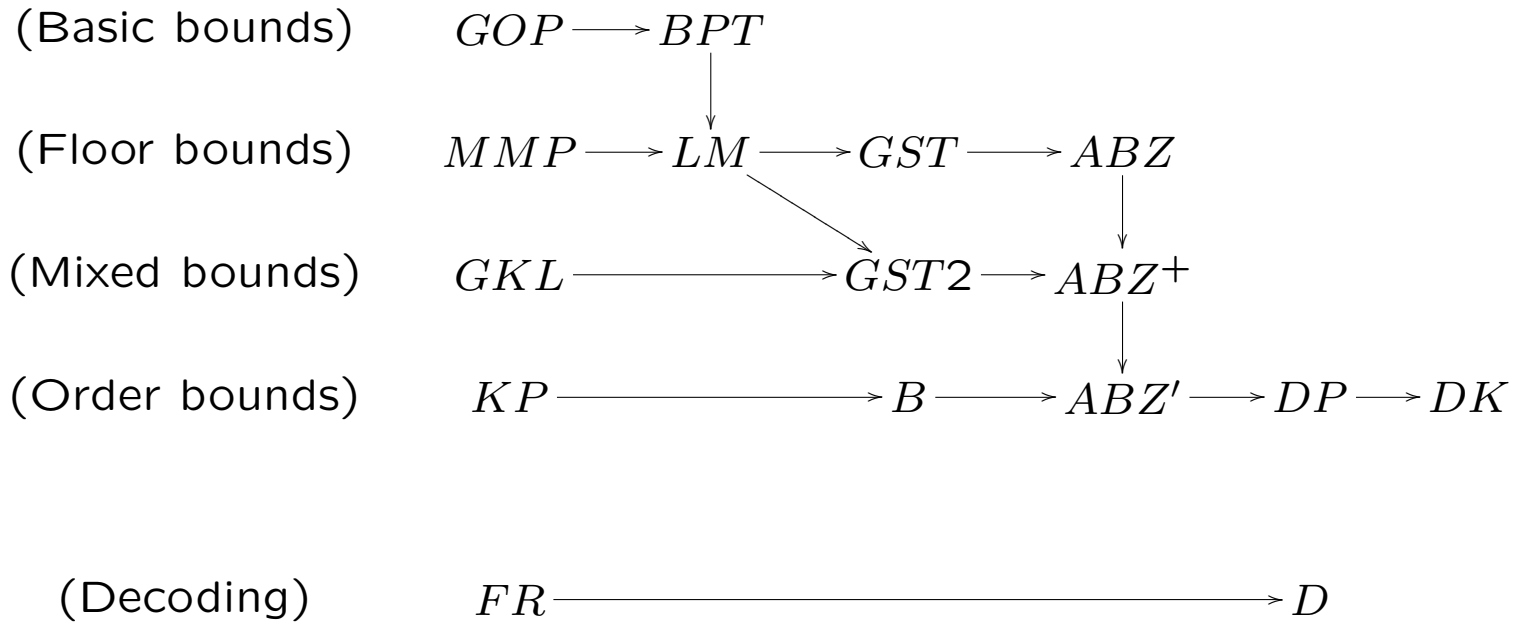
(2) D-Park'08a

(3) D-Kirov'09

Assumes	(1)	(2)	(3)
$\Delta \subseteq S'$	y	y	y
$S = \{1, 2, \ldots, n\}$	y	y	y
$P_i = P$, for $i \in \Delta$	y	y	
$P_i = P$, for $i = 1, 2, \ldots, n$	y		

4 - Comparison and Numerical results

Chart of known bounds



Code	d_{GST}	d_{GST2}	d_B	d_{ABZ}	d_{ABZ^+}	$d_{ABZ'}$
$C_\Omega(D, G = K + P + 2Q)$	8	8	7	8	8	8
$C_\Omega(D, G = K + 4P)$	7	6	8	7	7	8
$C_\Omega(D, G = K + 4P + Q)$	7	8	8	8	8	8
$C_\Omega(D, G = K + 4P + 2Q)$	9	9	9	10	10	10

Projective line over $\mathbb{F}_q$ (BCH bound)

For a cyclic code with zero set $I \supset \{1, \alpha, \ldots, \alpha^{m-2}\}$, $m \geq 2$, the minimum distance $d \geq m$.

	$C = mP$	
Δ	0 P $\vdots$ $(m-1)P$	$iP \in \Delta$ $\Leftrightarrow$ $i \geq 0$ and $i - m < 0$
S'	$\{P\}$	
Δ'		$iP \in \Delta'$ $\Leftrightarrow$ $i < 0$ and $i - m \geq 0$
S	$\emptyset$	
$\gamma \geq$	m	

Hermitian curve over $\mathbb{F}_{16}$ (Feng-Rao bound)

	$C = 3P$	$C = 4P$
Δ	0	0
	$4P$	$5P$
	$5P$	$10P$
	$9P$	$15P$
	$10P$	
	$14P$	
S'	$\{P\}$	$\{P\}$
Δ'	$3P$	
	$7P$	
	$11P$	
S	$\{P\}$	$\emptyset$
$\gamma \geq$	6	4

Suzuki curve over $\mathbb{F}_8$ ($C = 2P + 2Q$)

	(B)	(DP, DK)
Δ	0	0
	$8P$	$8P$
	$10P$	$10P$
	$13P$	$13P$
	$16P$	$16P + 2Q$
		$19P + 2Q$
	$21P$	$21P + 2Q$
	$29P$	$29P + 2Q$
S'	$\{P\}$	$\{P\}$
Δ'	$14P$	$14P$
	$15P$	$14P + Q^*$
	$27P$	$15P + Q$
		$15P + 2Q^*$
S	$\{P\}$	$\{P, Q\}$
$\gamma \geq$	7	8

Suzuki curve over $\mathbb{F}_8$.

$$C = C_\Omega(K - 5P + 8Q, D) \supset C_0 = C_\Omega(K - 4P + 8Q, D).$$

The Beelen bound yields $d(C_0) \geq 7$ and $d(C/C_0) \geq 6$. So that

$$d(C) \geq \min\{d(C/C_0), d(C_0)\} = 6.$$

Using $C_0 = C_\Omega(K - 5P + 9Q, D)$ produces the same numbers (since $13P \sim 13Q$). To improve the distance we only need to analyse words in the subset

$$C_\Omega(K - 5P + 8Q, D) \setminus C_\Omega(K - 4P + 8Q, D) \cup C_\Omega(K - 5P + 9Q, D).$$

The D-Kirov bound yields $\mathrm{wt}(c) \geq 8$ for such words. Thus $d(C) \geq 7$. The bound is sharp and is attained for words with support $A \sim 8Q - P \geq 8Q - 5P$.

Suzuki curve over $\mathbb{F}_8$ ($C = -5P + 8Q$)

	(B, DP)	(DK)
Δ	$10P - 3Q$	$10P - 3Q$
	$12P - 3Q$	$12P - 3Q$
	$13P - 3Q$	$13P - 3Q$
		$16P - 2Q^*$
		$16P - Q^*$
		$16P^*$
	$22P - 3Q$	$22P$
	$23P - 3Q$	$23P$
	$25P - 3Q$	
S'	$\{P\}$	$\{P, Q\}$
Δ'	$8P - 3Q$	$8P - 3Q$
	$16P - 3Q$	$16P - 3Q$
		$17P$
		$19P$
	$27P - 3Q$	$27P$
S	$\{P\}$	$\{P\}$
$\gamma \geq$	6	8

	Floor bounds		
	d_{LM}	d_{GST}	d_{ABZ}
d_{GOP}	6352	6352	6352
d_{LM}	·	2245	2852
d_{GST}	·	·	2213
d_{ABZ}	·	·	·
d_B	1	1	1
$d_{ABZ'}$	·	·	·
d_{GOP}	8	13	21
d_{LM}	·	7	15
d_{GST}	·	·	8
d_{ABZ}	·	·	·
d_B	1	1	1
$d_{ABZ'}$	·	·	·

Number of improvements of one bound over another (top), and the maximum improvement (bottom), based on 10168 two-point codes for the Suzuki curve over $\mathbb{F}_{32}$.

	Order bounds		
	d_B	d_{DP}	d_{DK}
d_{GOP}	6352	6352	6352
d_{LM}	4729	4731	4757
d_{GST}	4729	4731	4757
d_{ABZ}	4683	4685	4711
d_B	·	236	1565
d_{DP}	·	·	1366
d_{GOP}	33	33	33
d_{LM}	28	28	28
d_{GST}	24	24	24
d_{ABZ}	24	24	24
d_B	·	5	6
d_{DP}	·	·	6

Number of improvements of one bound over another (top), and the maximum improvement (bottom), based on 10168 two-point codes for the Suzuki curve over $\mathbb{F}_{32}$.

Joint work with Radoslav Kirov and Seungkook Park (KIAS)

- Coset bounds for algebraic geometric codes (DP'08a)
- Delta sets for divisors supported in two points (DP'08b)
- An extension of the order bound for AG codes (DK'09)
- Distance bounds for algebraic geometric codes (DKP'09)

- `http://www.math.uiuc.edu/{~duursma,~rkirov2}`
- `http://arxiv.org`
- `http://agtables.appspot.com`

Conclusions / Open problems

General theorems that cover all known bounds.

One of the bounds contains all other known bounds.

Bounds depend heavily on knowledge of multipoint non-gap structure.

Efficient implementations of the bounds available in python, numerical results available online.

Possible extensions: Improved codes, Multipoint codes, Bounds for generic error correction.

Question: What is the maximum $\gamma_\infty(C)$ of $\{\gamma(C; S, S')\}$ taken over all finite sets S, S'.