

Linear programming for secret sharing thresholds

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Linear codes and
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Let C be a code of length $n + 1$ with coordinates $\{1, \dots, n\} \cup \{0\}$. For a word (c_1, \dots, c_n, c_0) , the values c_1, \dots, c_n are shares for the secret c_0 .

Let d be the minimum distance of C and let d^\perp be the minimum distance of C^\perp .

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Let d be the minimum distance of C and let d^\perp be the minimum distance of C^\perp .

(Rejection) No party of size $\leq d^\perp - 2$ can reconstruct the secret from its shares.

(Acceptance) Every party of size $\geq n + 2 - d$ can reconstruct the secret uniquely.

$$\underbrace{0 \ \dots \ d^\perp - 2}_{\text{Rejected}} \quad \underbrace{\dots}_{?} \quad \underbrace{n + 2 - d \ \dots \ n}_{\text{Accepted}}$$

Let $C_0 \subset C$ be the subcode of words that are zero in position 0. Let $D = C^\perp$, and let $D_0 \subset D$ be the subcode of words that are zero in position 0.

Words in D_0 play no role in the rejection bound, they can not be used to reconstruct the secret.

Words in C_0 in C_0 play no role in the acceptance bound, they correspond to the secret value $c_0 = 0$ and they can not cause an ambiguity about the secret value even if they contain many zeros.

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(Rejection) No party of size $\leq d(D/D_0) - 2$ can reconstruct the secret from its shares.

(Acceptance) Every party of size $\geq n + 2 - d(C/C_0)$ can reconstruct the secret uniquely.

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For secret sharing we need a pair of dual codes C and D of length $n + 1$ such that words that are nonzero in the position 0 are of high weight in both the code and its dual.

Words that are zero in the last position have no role in the reconstruction of the secret (but they could be used to recover shares of other players or to test shares for consistency).

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For the projections of

$C = C_0 \cup (C \setminus C_0), D = D_0 \cup (D \setminus D_0)$ on

$\mathcal{P} = \{1, 2, \dots, n\}$, write

$$\begin{aligned} A &= C_0|_{\mathcal{P}}, & B &= (C \setminus C_0)|_{\mathcal{P}}. \\ X &= D_0|_{\mathcal{P}}, & Y &= (D \setminus D_0)|_{\mathcal{P}}. \end{aligned}$$

With weight enumerators $\mathbf{A}, \mathbf{B}, \mathbf{X}, \mathbf{Y}$, such that

$$\begin{aligned} W(C_0)(x, y) &= \mathbf{A}x, & W(C)(x, y) &= \mathbf{A}x + \mathbf{B}y. \\ W(D_0)(x, y) &= \mathbf{X}x, & W(D)(x, y) &= \mathbf{X}x + \mathbf{Y}y. \end{aligned}$$

$$\underbrace{0 \ \dots \ d(Y) - 1}_{\text{Rejected}} \quad \underbrace{\dots}_{?} \quad \underbrace{n + 1 - d(B) \ \dots \ n}_{\text{Accepted}}$$

Let

$$C(\Pi_1) = \left(\frac{1}{X_1} \mid \frac{1}{0} \right) \quad C(\Pi_2) = \left(\frac{1}{X_2} \mid \frac{1}{0} \right)$$

represent $\Sigma(\Pi_1)$ and $\Sigma(\Pi_2)$. Then,

$$C(\Pi) = \left(\frac{1}{X_1 \otimes 1} \mid \frac{1}{0} \right)$$
$$\left(\frac{I \otimes X_2}{0} \right)$$

represents $\Sigma(\Pi) = \Sigma(\Pi_1) \circ \Sigma(\Pi_2)$.

For a $(2, 3) \circ (3, 5)$ threshold scheme, use

$$f(x, y) \in \langle (1, x), (1, x, x^2)y, (1, x, x^2)y^2 \rangle.$$

For a $(3, 5) \circ (2, 3)$ threshold scheme, use

$$f(x, y) \in \langle (1, y, y^2), (1, y, y^2, y^3, y^4)x \rangle.$$

	b_1	b_2	b_3	b_4	b_5	\vdots		b_1	b_2	b_3	b_4	b_5
a_1	\cdot	\cdot	\cdot	\cdot	\cdot	\vdots	a_1	\cdot	\cdot		\cdot	
a_2	\cdot				\cdot	\vdots	a_2	\cdot			\cdot	
a_3	\cdot	\cdot				\vdots	a_3	\cdot		\cdot	\cdot	\cdot

Hermitian two-point codes

Theorem (Homma - Kim 2006, reformulation Seung Kook Park 2007)

Let $G = K + aP_\infty + bP_0 \geq K + P_\infty + P_0$, and write

$$a = a_0(q + 1) - a_1, \quad 0 \leq a_1 \leq q,$$

$$b = b_0(q + 1) - b_1, \quad 0 \leq b_1 \leq q.$$

Let $d = d(C_\omega(G, D))$, $d^* = \deg(G) - (2g - 2) = a + b$.

- | | | |
|------|---|---------------------------------------|
| (1) | $a_1, b_1 \leq a_0 + b_0,$ | $d = d^*.$ |
| (2a) | $b_1 \leq a_0 + b_0 \leq a_1,$ | $d = d^* + a_1 - (a_0 + b_0).$ |
| (2b) | $a_1 \leq a_0 + b_0 \leq b_1,$ | $d = d^* + b_1 - (a_0 + b_0).$ |
| (3a) | $a_0 + b_0 \leq a_1 \leq b_1, a_1 < q,$ | $d = d^* + a_1 + b_1 - 2(a_0 + b_0).$ |
| (3b) | $a_0 + b_0 \leq b_1 \leq a_1, b_1 < q$ | $d = d^* + a_1 + b_1 - 2(a_0 + b_0).$ |
| (4) | $a_0 + b_0 \leq a_1 = b_1 = q$ | $d = d^* + q - (a_0 + b_0).$ |

Bombieri's inner product on polynomials f and g is the following:

$$[f, g] = f\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)g(x_1, \dots, x_n)$$

If σ a unitary change of variables, $[f_\sigma, g_\sigma] = [f, g]$
On homogeneous weight enumerators, $\gamma = q - 1$:

$$\sigma = \begin{pmatrix} 1 & \gamma \\ 1 & -1 \end{pmatrix}$$

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Example for binary self-dual codes:

$$[f, g_\sigma - 2^n g] = 0$$

Choose

$$g = (x + y)^{k-2} y^{k+2} - 3(x + y)^{k-1} y^{k+1} + \\ + 6(x + y)^{k-1} y^{k+1} - 4(x + y)^{k-2} y^{k+2}.$$

Then $d < (1 - \frac{1}{q}) \frac{n}{2}$.

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Table for $n = n(dB, dY)$

Table for $dB = dB(dX, dY)$

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Let $C_1 \subseteq C_2$ be codes of length n with codimension 1.

Codes:

$$\begin{array}{cccc} C_1 & \subseteq & C_2 & C_2 \setminus C_1 \\ C_1^\perp & \supseteq & C_2^\perp & C_1^\perp \setminus C_2^\perp \end{array}$$

Weight Enumerators:

$$\begin{array}{ccc} \mathbf{A} & \mathbf{A} + \mathbf{B} & \mathbf{B} \\ \mathbf{X} + \mathbf{Y} & \mathbf{X} & \mathbf{Y} \end{array}$$

$2(n + 1)$ variables - **A**, **B**

$$\mathbf{A} \begin{matrix} = \\ \geq \\ \geq \\ \vdots \\ \geq \end{matrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\mathbf{X} = P_n \mathbf{A} + P_n \mathbf{B}$$

$$= \begin{pmatrix} |A + B| \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\mathbf{B} \begin{matrix} = \\ \vdots \\ = \\ \geq \\ \vdots \\ \geq \end{matrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \left. \vphantom{\begin{matrix} = \\ \vdots \\ = \\ \geq \\ \vdots \\ \geq \end{matrix}} \right\} d(B)$$

$$\mathbf{Y} = (q - 1)P_n \mathbf{A} - P_n \mathbf{B}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \left. \vphantom{\begin{matrix} = \\ \vdots \\ = \\ \geq \\ \vdots \\ \geq \end{matrix}} \right\} d(Y)$$

Table for $n = n(dB, dY)$

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$q = 2$

$dY \setminus dB$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	*	4	6	8	10	12	14
3	*	*	7	10	11	14	15
4	*	*	*	12	14	16	19
5	*	*	*	*	15	18	20
6	*	*	*	*	*	20	22
7	*	*	*	*	*	*	23

Table for $dB = dB(dX, dY)$

$q = 2, n = 13$

$dX \backslash dY$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	13	6	5	4	3	2	1	1	1	1	1	1	1
2	13	6	5	4	3	2	1	1	1	1	1	1	1
3	7	6	5	4	3	2	1	1	1	1	1	1	1
4	7	6	5	4	3	2	1	1	1	1	1	1	1
5	5	5	5	4	3	2	1	1	1	1	1	1	1
6	5	5	5	4	3	2	1	1	1	1	1	1	1
7	4	3	3	3	3	2	1	1	1	1	1	1	1
8	3	3	3	3	3	2	1	1	1	1	1	1	1
9	3	3	3	2	2	2	1	1	1	1	1	1	1
10	3	3	3	2	2	2	1	1	1	1	1	1	1
11	3	2	2	2	2	2	1	1	1	1	1	1	1
12	3	2	2	2	2	2	1	1	1	1	1	1	1
13	2	2	2	2	2	2	1	1	1	1	1	1	1

With respect to a special dot product:

$$\frac{q^{n+1}}{AA'} ([A, A'] + \frac{[B, B']}{q-1}) = [X, X'] + \frac{[Y, Y']}{q-1}$$

Dual LP:

$$A' \geq 0, A'_0 = 1$$

$$B' \geq 0, \text{ on } [a_1, n]$$

$$X' \leq 0$$

$$Y' \leq 0, \text{ on } [a_2, n]$$

Existence of a solution to the Dual LP, means
non-existence of a a_1, a_2 LSSS.

$$[f_\sigma, g_\sigma] = q^{n+1}[f, g]$$

LSSS split weight enumerators:

$$f_{A,B}(x, y, u, v) = f_A(x, y)u + f_B(x, y)v$$

$$f_\sigma = f_X u + f_Y v$$

"Singleton Bound" Example:

$$g(x, y, u, v) = (x + \gamma y)^k x^{n-k} (u - v)$$

$$g_\sigma = g(x + \gamma y, x - y, u + \gamma v, u - v) = -q^{k+1} (x + \gamma y)^{n-k} x^k v$$

if $dB \geq k + 1$ and $dY \geq n - k + 1$, contradiction! thus

$$dB \leq n + 1 - dY$$

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Connections with other coding theory problems:

1. $\max d = d(A + B)$ in terms of $d' = d(X + Y)$.
2. $\max \rho = d(B)$ in terms of $d' = d(X + Y)$.
3. $\max a_1 = d(B)$ in terms of $a_2 = d(Y)$.