INTERSECTION MULTIPLICITY ON BLOW-UPS

S. P. DUTTA
(dutta@math.uiuc.edu)

ABSTRACT. A conjecture on vanishing and non-negativity of intersection multiplicity on the blow-up of a regular local ring at its closed point has been proposed. Proofs of vanishing, several special cases of non-negativity and a sufficient condition for non-negativity of this conjecture are described. The topic on intersection multiplicity on vector bundles is also addressed.

Introduction.

Let $X$ be a regular scheme of finite type over a field or an excellent discrete valuation ring and $\mathcal{F}, \mathcal{G}$ be two coherent $O_X$-modules such that $\ell(H^i(X, \text{Tor}_j^{O_X}(\mathcal{F}, \mathcal{G}))) < \infty$, for $i, j \geq 0$. We define $\chi^{O_X}(\mathcal{F}, \mathcal{G})$ as $\Sigma(-1)^{i+j}\ell(H^i(X, \text{Tor}_j^{O_X}(\mathcal{F}, \mathcal{G})))$. Note that this is an extension of Serre’s definition of intersection multiplicity on regular local rings ([Se]).

First, we would propose the following conjecture:

Conjecture. Let $(R, m, K)$ be a regular local ring of dimension $n$ and of essentially finite type over a field or an excellent discrete valuation ring. Let $\tilde{X}$ denote the blow-up of $X = \text{Spec} R$ at the closed point $s = [m]$. Let $\tilde{Y}, \tilde{Z}$ be any two subvarieties of $\tilde{X}$ such that $\tilde{Y} \cap \tilde{Z} \subset \mathcal{E} = \mathbb{P}^{n-1}_K$. The following statements hold:

1) if $\dim \tilde{Y} + \dim \tilde{Z} < \dim \tilde{X}$, then $\chi^{O_{\tilde{X}}}(O_{\tilde{Y}}, O_{\tilde{Z}}) = 0$

2) if $\dim \tilde{Y} + \dim \tilde{Z} = \dim \tilde{X}$ and if at least one of $\tilde{Y}$ and $\tilde{Z}$ is not contained in $\mathcal{E}$, then $\chi^{O_{\tilde{X}}}(O_{\tilde{Y}}, O_{\tilde{Z}}) \geq 0$.

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The study of this conjecture is the central objective of this paper. In this article we prove the following (statement of Theorem towards the end of the introduction):

1. **Validity of part i)** i.e., vanishing part of the conjecture in any characteristic.

2. **For the equicharacteristic case:** validity of the conjecture when $\tilde{Y}, \tilde{Z}$ are the blow-ups of $\text{Spec}(R/P)$ and $\text{Spec}(R/q)$ respectively.

3. **Several special cases of part (ii) of the above Conjecture.**

4. **A rather surprising (to us!) correlation between multiplicity of each component of the closed fiber over $s = [m]$ in a regular alteration of a particular faithfully flat extension of $\tilde{Y}(\tilde{Z})$ and non-negativity of $\chi^O_{\tilde{X}}(O_{\tilde{Y}}, O_{\tilde{Z}})$ is established in the following way:** if the multiplicity is 1 at each component, then $\chi \geq 0$; if the multiplicity is $> 1$ at any such component then a sufficient condition for non-negativity is derived for the corresponding component.

We would like to mention here that even in equicharacteristic our conjecture is open in the general set-up i.e., when at least one of $\tilde{Y}$ and $\tilde{Z}$ is not a blow-up of $\text{Spec}(R/P)(\text{Spec}(R/q))$ at its closed point. If $\tilde{X}$ is smooth over the ground field, the vanishing part i.e., part i of our conjecture can be derived from Fulton-MacPherson’s result stated later in this introduction; however part ii of the conjecture is completely open even in this case. From the point of view of our approach we could see that there is a potential for failure of non-negativity at a certain component of the normal cone $C$ to $Y' \cap Z' \hookrightarrow Y'$, where $Z'$ is a regular alternation of $\tilde{Z}$, $Z' \hookrightarrow X' = \mathbb{P}^N_X$ is a closed imbedding and $Y'$ is the pull back of $\tilde{Y}$ in $X'$, provided this component is contained in the intersection of two components of the normal bundle to $Z' \hookrightarrow X'$. We were unsure of: a) whether the above situation can be avoided, b) whether there exists a necessary and sufficient condition for non-negativity—especially at any component of the normal cone $C$ mentioned above and c) whether there exists any significant case of non-negativity in the most general set-up. Search for affirmative answers to these questions along with a proof of the vanishing part constitute the main thrusts of this paper.

Before providing a synopsis of our approach, let me first state Serre’s conjecture on intersection multiplicity for regular local rings, the starting point and the motivating force
for all such work, along with a short account of past important work on this conjecture. We hope this will help the reader to appreciate the evolution of the conjecture in proper perspective.

**Serre’s conjecture and a brief history.**

**Conjecture (Serre) (Ch. V, [Se]).** Let $R$ be a regular local ring and let $M, N$ be two finitely generated $R$-modules such that $\ell(M \otimes_R N) < \infty$. Let $\chi(M, N) = \Sigma(-1)^i\ell(\text{Tor}_i^R(M, N))$. Then $\chi(M, N) \geq 0$, the sign of equality holds if and only if $\dim M + \dim N < \dim R$.

This conjecture can be divided into three parts:

(a) Vanishing: $\chi(M, N) = 0$ when $\dim M + \dim N < \dim R$,

(b) Non-negativity: $\chi(M, N) \geq 0$,

and (c) Positivity: $\chi(M, N) > 0$ when $\dim M + \dim N = \dim R$.

Serre proved his conjecture for equicharacteristic and unramified regular local rings (Ch. V, [Se]). He also proved that for $R, M, N$ as above, if $\ell(M \otimes_R N) < \infty$, then $\dim M + \dim N \leq \dim R$. Moreover, he showed that in the equicharacteristic case, when $\dim M + \dim N = \dim R$, $\chi(M, N) \geq e_m(M)e_m(N)$.

P. Roberts [R1], H. Gillet and C. Soulé [G-S] proved the vanishing part independently (in mid-eighties). Their proofs extend to the local complete intersections when both $M$ and $N$ have finite projective dimension. The hope of generalizing the validity of this conjecture to non-regular rings when only one of the modules has finite projective dimension was dashed by a counterexample due to Hochster, McLaughlin and this author [D-H-M] in the early eighties. This example also led to counter-examples to several other multiplicity related conjectures. In the mid-nineties Gabber [B] proved the non-negativity part of the conjecture. The positivity part of Serre’s conjecture has been open for more than fifty years. The fact that positivity or non-negativity implies vanishing was proved in [D1] in the early eighties.

In this context we would also mention the following result due to Fulton and MacPherson in the mid-seventies ([Fu-Ma]): If $X$ is a smooth variety over a field, $Y$ and $Z$ are two closed subvarieties of $X$ and $m = \dim Y + \dim Z - \dim X$, then $\tau_k(\text{Tor}_X^Y(Y, Z)) = 0$ for
In [D7] we proved a result that connects Serre’s conjecture with our conjecture stated earlier in the following way.

Let $R$ be a regular local ring of essentially finite type over a discrete valuation ring or a field, and let $P, q$ be two prime ideals such that $\ell(R/(P+q)) < \infty$. Let $\tilde{X}, \tilde{Y}, \tilde{Z}$ denote the blow-ups of $\text{Spec } R$, $\text{Spec}(R/P)$, $\text{Spec}(R/q)$ respectively at the respective closed points; let $P_{\chi(G/R/P),G(R/q)}$ denote the alternating sum of Hilbert polynomials $P_{\text{Tor}_i^{G(R/P),G(R/q)}}$, where $G(T)$ denotes the graded module corresponding to the $m$-adic filtration on a finitely generated $R$-module $T$. For any finitely generated $R$-module $M$, let $e_m(M)$ denote the Hilbert-Samuel multiplicity of $M$ with respect to the maximal ideal $m$.

**Theorem 1** (Th. 2, [D7]). With notations as above, we have:

$$
\chi(R/P, R/q) = P_{\chi(G(R/P),G(R/q))} + \chi^{O_{\tilde{X}}}(O_{\tilde{Y}}, O_{\tilde{Z}}).
$$

The above theorem leads to a new approach to Serre’s conjecture as a whole. It shows that the validity of our conjecture for a more restricted set-up implies a stronger version of Serre’s conjecture i.e., vanishing and $\chi(M, N) \geq e_m(M)e_m(N)$ when $\dim M + \dim N = \dim R$ (see Corollary below). The proof of this theorem uses filtered resolutions ([Se]) and an extension of a theorem of Peskine and Szpiro ([P-S2]).

As a corollary, the following is deduced from the above theorem:

i) if $\dim R/P + \dim R/q < \dim R$, then $\chi(R/P, R/q) = \chi^{O_{\tilde{X}}}(O_{\tilde{Y}}, O_{\tilde{Z}})$ and

ii) if $\dim R/P + \dim R/q = \dim R$, then $\chi(R/P, R/q) = e_m(R/P)e_m(R/q) + \chi^{O_{\tilde{X}}}(O_{\tilde{Y}}, O_{\tilde{Z}})$.

A different version of part ii of this corollary has been presented in Fulton’s book “Intersection Theory” (Example 20.4.3, [Fu]). The suggested technique for proof of this version in ([Fu]), involving local chern characters and the Riemann-Roch theorem, is completely different from our proof. Moreover, it does not shed any light when $R/P$ and $R/q$ do not intersect properly (part i of the above Corollary) (recall that Serre-vanishing was not known at the time of publication of [Fu], Example 20.4.1 [Fu]).
Due to the results of Serre \[S1\], Roberts \[R1\] and Gillet-Soulé \[G-S\] mentioned earlier, the above corollary implies the following.

**Result.** Let $\tilde{X}, \tilde{Y}, \tilde{Z}$ be the respective blow-ups as above. If $R$ is equicharacteristic, our conjecture is valid and if $R$ has mixed characteristic, part i) of the conjecture is valid.

Now we would like to describe, in a very brief manner, the approach/idea behind the proof of results 1., 2., 3., and 4. mentioned earlier.

In our attempt to understand the behavior of $\chi$ on blow-ups and for that matter on regular schemes we appeal to Gabber’s idea ([B], [G], [Ho2]) and to several results from “Intersection theory” by Fulton ([Fu]). The standard techniques for proving non-negativity of the intersection product in equicharacteristic (Ch. 12, Ch. 18 in [Fu]) do not seem to be of much help in mixed characteristic. In the mid-nineties Gabber came up with a brilliant idea to prove the non-negativity part of Serre’s conjecture. We apply his method of using a) de Jong’s theorem on regular alterations ([J]) and b) generalization of a spectral sequence argument of Serre to reduce the question on $\chi$ over a regular scheme to a similar question on vector bundles over projective schemes arising out of regular imbeddings. Using ramification of $R$ and the module of differentials, Gabber showed in a highly ingenious way that when the regular scheme is Spec of a ramified/equicharacteristic regular local ring, the fiber over the closed point of the vector bundle resulting from the technique mentioned above is generated by global sections. This fact played a crucial role in his proof of non-negativity. *Unfortunately, over the blow-up of a regular local ring, the corresponding bundle, induced by a similar technique, is not generated by global sections—actually it is not even nef (part v our of Theorem).*

Results in section 2, [D7] were developed to bypass this difficulty to a considerable extent. These results constitute key ingredients for our proof of our theorem on blow-ups in this paper. Hence we mention them here very briefly. The first result (Theorem 2) describes intersection multiplicity of a closed subvariety $V$ of a vector bundle $E$ on a projective scheme $W$ over a field with respect to the 0-section in terms of the Riemann-Roch Theorem ([Fu]). We will use this theorem several times in our work. It shows that for vanishing we do not need $E$ to be generated by global sections.
Theorem 2 (Prop. 2.2, [D7]). Let $W$ be a projective scheme over a field and let $\mathcal{L}$ be a locally free $\mathcal{O}_W$-module of rank $d$. Let $E = \operatorname{Spec} \left( \operatorname{Sym}_{\mathcal{O}_W}(\mathcal{L}) \right)$ and let $V$ be a closed subscheme of $E$. Let $\beta : W \to E$ be the 0-section; we identify $W$ with $\beta(W)$ when there is no scope of ambiguity. We have the following:

i) $\chi^{\mathcal{O}_E}(\mathcal{O}_V, \mathcal{O}_W) = \int_{\mathcal{V}} c_d(\xi_{\mathcal{V}})td(\xi_{\mathcal{V}})^{-1} \cap \tau(\mathcal{O}_{\mathcal{V}})$, where $\xi_{\mathcal{V}}$ is the restriction of the universal quotient bundle $\xi$ on $\mathbb{P}(E \oplus 1)$ to the projective closure $\mathcal{V}$ of $V$;

ii) if $\dim V < d$, then $\chi^{\mathcal{O}_E}(\mathcal{O}_V, \mathcal{O}_W) = 0$;

iii) if $V \cap W = \phi$, then $\chi^{\mathcal{O}_E}(\mathcal{O}_V, \mathcal{O}_W) = 0$; and

iv) if $\dim V = d$, then $\chi^{\mathcal{O}_E}(\mathcal{O}_V, \mathcal{O}_W) = \int_{\mathcal{V}} c_d(\xi_{\mathcal{V}})[\mathcal{V}] = \int \beta^*([V])$.

This theorem plays an important role in our proof of the vanishing part of the conjecture. Moreover, part iv above, along with Theorem 12.1(a) in [Fudem] demonstrates a new proof of non-negativity of $\chi^{\mathcal{O}_E}(\mathcal{O}_V, \mathcal{O}_W)$ when $E$ is generated by global sections ([G], [B], [Ho2]).

In the final proposition and its corollary of section 2 in [D7] we described the effect of $\chi$ via the pull-back map between two vector bundles in the following way:

Corollary. Let $X$ be a smooth projective variety over a field $K$ such that $T_X$ is generated by global sections. Let $W$ be a smooth closed subvariety of codimension $r$. Let $\mathcal{I}$ denote the ideal of definition of $W$ in $X$ and let $E$ be the corresponding normal cone. Let $E'$ be a vector bundle on $W$ and let $V$ be a closed subvariety of $E'$. Suppose that there exists a map $\varphi : E \to E'$ of bundles such that the dimension of support of $\operatorname{Tor}^{\mathcal{O}_E}_{i}(\mathcal{O}_V, \mathcal{O}_E) < r$ for $i > 0$. Then, for any coherent sheaf $\mathcal{F}$ on $W$, $\chi^{\mathcal{O}_{E'}}(\mathcal{O}_V, \mathcal{F}) = \chi^{\mathcal{O}_E}(\mathcal{O}_{\varphi^{-1}(V)}, \mathcal{F})$. Moreover, if $\dim \varphi^{-1}(V) \leq r$, then $\chi^{\mathcal{O}_{E'}}(\mathcal{O}_V, \mathcal{O}_W) \geq 0$.

This corollary has been used several times in the proofs of parts iv and v of our theorem in this paper.

In section 1, as mentioned earlier, we prove our theorem on intersection multiplicity on blow-ups. Our approach actually works in any characteristic. In order to understand $\chi$ in the mixed characteristic, when $\dim \tilde{Y} + \dim \tilde{Z} = \dim \tilde{X}$ and neither $\tilde{Y}$ nor $\tilde{Z}$ is contained in the exceptional divisor, we use the semi-stability part of de Jong’s theorem on regular
alterations ([J]) and several aspects of intersection theory from [Fu]. We state some of these results in the beginning of Section 1. To describe our final theorem (in particular for parts iv and v) we need the following set-up.

Let $R$ be a regular local ring essentially of finite type over a field or over a complete discrete valuation ring such that the residue field of $R$ is perfect. Let $Z'$ be a regular alteration of $\tilde{Z}$. Then $Z'$ has a closed immersion in $\mathbb{P}^{N'}_X = X'$ for some $N' > 0$. Let $\pi' : X' \to \tilde{X}$ be the projection map and let $Y' = \pi'^{-1}(\tilde{Y})$. Now repeat this process for $Y'$ and obtain $Y'' \hookrightarrow X''$, where $Y''$ is a a regular alteration of $Y'$, $X'' = \mathbb{P}^{N''}_{X''}$, $\pi'' : X'' \to X'$ is the projection map and $Z'' = \pi''^{-1}(Z')$. Then $Y'', Z''$ are regular closed subschemes of the regular scheme $X''$. Let $\mathcal{I}$ and $\mathcal{J}$ be the ideals of definitions of $Y''$ and $Z''$ respectively in $\mathcal{O}_{X''}$; let $E = \text{Spec}(\text{Sym}_{\mathcal{O}_{Y''}}(\mathcal{I}/\mathcal{I}^2))$—the normal bundle to the regular imbedding $Y'' \hookrightarrow X''$ and let $V = \text{Spec}(\bigoplus_{t \geq 0} \mathcal{I}^t \mathcal{O}_{Z''}/\mathcal{I}^{t+1} \mathcal{O}_{Z''})$—the normal cone to $Y'' \cap Z''$ in $Z''$. Let $E_1, \ldots, E_h; V_1, \ldots, V_e$ and $W_1, \ldots, W_h$ denote the components of $E_s$ (fiber over $s = [m]$), $V$ and $Y''_s$ respectively; also let $m_i$ denote the multiplicity of $W_i$ in $Y''_s$ for $1 \leq i \leq h$. We have a set map $\sigma : [1, e] \to [1, d]$ such that $V_i \subset E_{\sigma(i)}$ for $1 \leq i \leq e$.

Application of two regular alterations, once on $\tilde{Z}$ and next on the pull back of $\tilde{Y}$ in $X'$ are performed in such a way that we can avoid the situation where a component of the normal cone to $Y' \cap Z'$ in $Y'$ might be contained in the intersection of two components of (the normal cone to $Z' \hookrightarrow X'$) $\times_{Z'} Z''_s$. Such a situation creates an impediment in the way of ascertaining non-negativity via our approach.

Basically parts iv and v of our theorem prove that when multiplicity of each component of $Y''_s$ in $Y'$ is 1, then desired non-negativity for our conjecture is achieved. However, if multiplicity of a certain component $W_i$ is greater than 1, non-negativity is achieved if the corresponding component $V_i$ of the cone $V$ to $Y'' \cap Z'' \hookrightarrow Y''$ is not contained in a specific sub-bundle $F_i$ of $E'_i$, the restriction of $E$ to $W_i$.

The statement of our theorem is the following:

**Theorem.** With notations as above, we have the following:

(i) if $\dim \tilde{Y} + \dim \tilde{Z} < \dim \tilde{X}$, then $\chi^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) = 0$.

Assume henceforth $\dim \tilde{Y} + \dim \tilde{Z} = \dim \tilde{X}$. 

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(ii) if both $\tilde{Y}$ and $\tilde{Z}$ are contained in $E$—the exceptional divisor, then $\chi^{O_X}(O_{\tilde{Y}}, O_{\tilde{Z}}) < 0$; if only one of them is contained in $E$, then $\chi^{O_X}(O_{\tilde{Y}}, O_{\tilde{Z}}) > 0$.

Henceforth assume that neither $\tilde{Y}$ nor $\tilde{Z}$ is contained in $E$.

(iii) if $\dim(G(R/P) \otimes G(R/q)) \leq 1$, then $\chi^{O_X}(O_{\tilde{Y}}, O_{\tilde{Z}}) \geq 0$, where $\tilde{Y}$, $\tilde{Z}$ are the blow-ups of $\text{Spec}(R/P)$ and $\text{Spec}(R/q)$ respectively at the closed point.

(iv) if $m_{\sigma(i)} = 1$ for some $i$, then $\chi^{O_{E_{\sigma(i)}}}(O_{W_{\sigma(i)}}, O_{V_i}) \geq 0$; if $m_{\sigma(i)} = 1$ for $1 \leq i \leq e$, then $\chi^{O_X}(O_{\tilde{Y}}, O_{\tilde{Z}}) \geq 0$.

(v) Suppose that $m_{\sigma(i)} > 1$ for some $i$. Then there exists a sub-bundle $F_i$ of $E_{\sigma(i)}$ such that a) if $V_i \not\subset F_i$, then $\chi^{O_{E_{\sigma(i)}}}(O_{W_{\sigma(i)}}, O_{V_i}) \geq 0$ and b) if $V_i \subset F_i$, then $\chi^{O_{E_{\sigma(i)}}}(O_{W_{\sigma(i)}}, O_{V_i}) \leq 0$.

As a final comment on the above theorem we should point out that by construction the sub-bundle $F_i$ is quite accessible and it is even generated by global sections.

It is clear that this theorem makes progress on the positivity part of Serre’s conjecture when additional hypothesis holds. As a corollary we derive a new proof of Serre-vanishing and several special cases of a stronger version of the positivity part of Serre’s conjecture.

This paper is a sequel to our previous work in [D7].

For notations and terminology in commutative algebra and algebraic geometry we refer the reader to [Ma] and [H] respectively. For the notions of chern class, local chern character, intersection product and related matters we refer the reader to [Fu]. By $\dim R$ (or $\dim X$) we mean the dimension of $R$ (or dimension of $X$). While studying intersection multiplicity on a regular local ring $(R, m, K)$ we can assume without any loss of generality that $K$ is algebraically closed and the modules $M, N$ can be replaced by $R/P$, and $R/q$ respectively where $P$ and $q$ are prime ideals in $R$. For any coherent sheaf $F$ on a scheme $X$, $\mathbb{P}(F)$ denotes the projective scheme $\text{Proj} \left( \text{Sym}_{O_X}(F) \right)$ and $F^\vee$ stands for $\text{Hom}_{O_X}(F, O_X)$.

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Section 1

We are going to investigate the behavior of $\chi^{O_Y}(O_Y, O_Z)$ for the mixed characteristic case. It can be checked easily that the arguments for the equicharacteristic case, described in the previous section, does not work in this situation. For proof of our theorem we need several results from different perspectives. We describe some of them below.

**Result 1** (Prop. 3.3 in [Fu]) Let $E$ be a vector bundle of rank $r$ on an algebraic scheme $X$, $\pi : E \to X$ the projection, $s = s_E$ denote the 0-section of this bundle. Let $\beta \in A_k E$, and let $\overline{\beta}$ be any element of $A_k(\mathbb{P}(E \oplus 1))$ which restricts to $\beta$ in $A_k E$. Then $s^*(\beta) = (\pi^*)^{-1}(\beta)$ where $q$ is the projection from $\mathbb{P}(E \oplus 1)$ to $X$ and $\xi$ is the universal rank $r$ quotient bundle of $q^*(E \oplus 1)$.

For a proof we refer the reader to [Fu].

**Result 2** (part (a) of Theorem 12.1 in [Fu]) Let $E$ be a vector bundle of rank $r$ on a scheme $X$, $\pi : E \to X$ the projection, $s_E : X \to E$ the 0-section. Let $V$ be a $k$-dimensional subvariety of $E$, $k \geq r$. If $E$ is generated by sections, then $s^*_E[V] \in A^k_{k-r}(X)$.

For a proof we refer the reader to [Fu].

**Result 3** (Statement of Theorem 6.5, [J]) Let $X$ be an $S$-variety ($S = \text{spec of a complete discrete valuation ring } v$) and let $Z \subset X$ be a proper closed subset with $f : X \to S$ such that $f^{-1}([m_v])$ (the unique closed point of $S$) $\subset Z$. Then there exists a trait $S_1 = \text{Spec } v_1$, $v_1$-a complete discrete valuation ring, finite over $S$, an $S_1$-variety $X_1$, an alteration of schemes $\phi_1 : X_1 \to X$ and an open immersion $j_1 : X_1 \to \overline{X}_1$ of $S_1$-varieties, with the following properties:

i) $\overline{X}_1$ is projective $S_1$-variety with geometrically irreducible generic fiber, and

ii) the pair $(\overline{X}_1, \phi_1^{-1}(Z)_{\text{red}} \cup \overline{X}_1 - j_1(X_1))$ is strictly semi-stable.

For a proof we refer the reader to [J].

The main reason for stating de Jong’s theorem in the mixed characteristic is that we are
going to use the stronger aspect i.e., the semi-stability property of alteration in our proof. We refer the reader to Theorem 4.1 in [J] for the statement in the equicharacteristic case.

Let us first describe the set-up. Let \( R \) be a regular local ring of essentially finite type over a field \( k \) or a complete discrete valuation ring \( v \) and let the residue field \( K \) of \( R \) be perfect. \( \tilde{X} = \text{blow-up of } X = \text{Spec } R \) at \( s = [m] \); \( \tilde{Y}, \tilde{Z} \) two closed subvarieties of \( \tilde{X} \) such that \( \tilde{Y} \cap \tilde{Z} \subset \mathcal{E} = \mathbb{P}^{n-1}_K \)—the exceptional divisor. Let \( Z' \) be a regular alteration of \( \tilde{Z} \). Then \( Z' \) has a closed immersion in \( \mathbb{P}^{N'}_{\tilde{X}} = X' \) for some \( N' > 0 \). Let \( \pi' : X' \to \tilde{X} \) be the projection and let \( Y' = \pi'^{-1}(\tilde{Y}) \). Let \( T' \) denote the defining ideal of \( Z' \) in \( X' \) and let \( r = \text{codim } Z' \in X' \). Repeat this process for \( Y' \) i.e., let \( Y'' \) be a regular alteration of \( Y' \). Then \( Y'' \) has a closed immersion \( \mathbb{P}^{N''}_{\tilde{X}} = X'' \) for some positive integer \( N'' > 0 \). Let \( \pi'' : X'' \to X' \) be the projection and let \( Z'' = \pi''^{-1}(Z') \). Let \( I, J \) denote defining ideals of \( Y'' \) and \( Z'' \) in \( X'' \) respectively. Let \( d = \text{codim } Y'' \in X'' \). Since \( Y'', Z'', X'' \) are all regular, \( I \) and \( J \) are locally generated by regular sequences of length \( d \) and \( r \) respectively. Write \( E = \text{Spec } (\text{Sym}_{\mathcal{O}_{Y''}}(I/I^2)) \) and \( V = \text{Spec } (\bigoplus_{t \geq 0} I^t \mathcal{O}_{Z''}/I^{t+1} \mathcal{O}_{Z''}) \). Let \( E_1, \ldots, E_h, V_1, \ldots, V_e \) and \( W_1, \ldots, W_h \) denote the components of \( E_s, V \) and \( Y'' \) (fiber over \( s \)) respectively; let \( m_i \) denote the geometric multiplicity of \( W_i \) in \( Y'' \) for \( 1 \leq i \leq h \). We have a set map \( \sigma : [1, e] \to [1, h] \) such that \( V_i \subset E_{\sigma(i)} \) for \( 1 \leq i \leq e \). Now we state our theorem.

**Theorem.** With notations as above, we have the following:

i) if \( \dim \tilde{Y} + \dim \tilde{Z} < \dim \tilde{X} \) or if \( \tilde{Y} \cap \tilde{Z} = \phi \), then \( \chi^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) = 0 \).

Assume \( \dim \tilde{Y} + \dim \tilde{Z} = \dim \tilde{X} \).

ii) if both \( \tilde{Y} \) and \( \tilde{Z} \) are contained in \( \mathcal{E} \)—the exceptional divisor, then \( \chi^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) < 0 \); if only one of \( \tilde{Y} \) and \( \tilde{Z} \) is contained in \( \mathcal{E} \), then \( \chi^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) > 0 \).

Henceforth assume that neither \( \tilde{Y} \) nor \( \tilde{Z} \) is contained in \( \mathcal{E} \).

iii) if \( \dim G(R/P) \otimes G(R/q) \leq 1 \) and \( \tilde{Y}, \tilde{Z} \) are the blow-ups of \text{Spec } (R/P), \text{Spec } (R/q) respectively at their closed points, then \( \chi^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) \geq 0 \).

iv) if \( m_{\sigma(i)} = 1 \) for some \( i \), then \( \chi^{\mathcal{O}_{E_{\sigma(i)}}}(\mathcal{O}_{V_i}, \mathcal{O}_{W_{\sigma(i)}}) \geq 0 \); if all \( m_{\sigma(i)} = 1 \) for \( 1 \leq i \leq e \), then \( \chi^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) \geq 0 \).

v) suppose \( m_{\sigma(i)} > 1 \) for some \( i \). Then there exists a sub-bundle \( F_i \) of \( E_{\sigma(i)} \) such that
a) if \( V_i \not\subset F_i \), then \( \chi^{O_{E_{(i)}}}(O_{W_{(i)}}, O_{V_i}) \geq 0 \) and b) if \( V_i \subset F_i \), then \( \chi^{O_{E_{(i)}}}(O_{W_{(i)}}, O_{V_i}) \leq 0 \).

**Proof.** i) Let \( \pi' : Z' \to \tilde{Z} \) be a regular alteration (Th. 4.1, Th. 6-5, [J]). We can assume \( Z' \hookrightarrow \text{Proj}(O_{\tilde{Z}}[U_1, \ldots, U_{N+1}]) \) and hence \( Z' \hookrightarrow X' = \text{Proj}(O_{\tilde{X}}[U_1, \ldots, U_{N+1}]) \) is a closed imbedding such that the diagram

\[
\begin{array}{c}
Z' \quad \hookrightarrow \quad X' \\
\uparrow \quad \quad \quad \uparrow \pi \\
\tilde{Z} \quad \hookrightarrow \quad \tilde{X}
\end{array}
\]

commutes. Since \( X' \) and \( Z' \) are both regular schemes, \( Z' \) is a local complete intersection in \( X' \) of codimension \( r \) say. Let \( I' \) denote the ideal of definition of \( Z' \) in \( X' \). Then \( I' \) is locally defined by a regular sequence of length \( r \).

Let \( Y' = \pi^{-1}(\tilde{Y}) \). Since \( X' \to \tilde{X} \) is flat of relative dimension \( N \), \( \dim Y' = \dim \tilde{Y} + N \).

Note that \( \dim Z' = \dim \tilde{Z} \) by construction. Hence \( \dim Y' + \dim Z' \leq \dim X' \); equality holds if and only if \( \dim \tilde{Y} + \dim \tilde{Z} = \dim \tilde{X} \). Since \( \pi \) is proper, by pulling back a locally free \( O_{\tilde{X}} \)-resolution of \( O_{\tilde{Y}} \) via \( \pi \), we obtain, by projection formula,

\[
\chi^{O_{X'}}(O_{Y'}, O_{Z'}) = \chi^{O_{\tilde{X}}}(O_{\tilde{Y}}, \pi_* O_{Z'}) + \sum_{i \geq 1} (-1)^i \chi^{O_{\tilde{X}}}(O_{\tilde{Y}}, R^i \pi_* O_{Z'})
\]

(1)

Since \( \dim (\text{support of } \pi_* O_{Z'}) = \dim (\text{support of } O_{\tilde{Z}}) \) \((Z' \to \tilde{Z} \text{ is an alteration})\) and for \( i > 0 \), \( \dim (\text{support of } R^i \pi_* O_{Z'}) < \dim (\text{support of } O_{\tilde{Z}}) \), by induction on dimension, it is enough to prove the theorem for \( \chi^{O_{X'}}(O_{Y'}, O_{Z'}) \).

Let \( E' = \text{Spec}(O_{X'}/I' \oplus I'/I'^2 \oplus \cdots) \). Then \( E' \) is a vector bundle of rank \( r \) over \( Z' \) and let \( w : Z' \to E' \) be the zero-section. Let \( V' = \text{Spec}(O_{Y'}/I' O_{Y'} \oplus I'/O_{Y'}/I'^2 O_{Y'} \oplus \cdots) \).

Then \( \dim E' = \dim Z' + r = \dim X' \) and \( \dim V' = \dim Y' \). By a spectral sequence argument due to Serre and its extension to schemes due to Gabber, we have

\[
\chi^{O_{X'}}(O_{Y'}, O_{Z'}) = \chi^{O_{E'}}(O_{V'}, O_{Z'})
\]

(2)

Note that, since \( \tilde{Y} \cap \tilde{Z} \subset \mathcal{E}, \exists t > 0 \), such that \( m^t O_{V'} = 0 \). Hence by considering a filtration \( O_V \supset m O_V \supset \cdots \supset m^{t-1} O_V \supset 0 \), we can replace \( O_{V'} \) by \( \mathcal{G} \) such that \( m \cdot \mathcal{G} = 0 \).
Let $s$ denote the closed point of $X$. Let $X'_s$, $E'_s$ and $Z'_s$ denote the fibers of $X'$, $E'$ and $Z'$ over $s$ respectively. Let $Z'_1, \ldots, Z'_g$ be the irreducible components of $Z'_s$. Let $E'_i$ be the restriction of $E'_s$ to $Z'_i$. Then $E'_1, \ldots, E'_g$ are the irreducible components of $E_s$. By induction on dimension, we can replace $G$ by an irreducible component $V'_i$ of the support of $G$, contained in $E_i$ for some $i$, $1 \leq i \leq e$. We have:

$$
\chi^{O_{E'}}(O_{V'_i}, O_{Z'_i}) = \chi^{O_{E'_i}}(O_{V'_i}, O_{Z'_i}) = \chi^{O_{E'_i}}(O_{V'_i}, O_{Z'_i}).
$$

(3)

If $\dim \tilde{Y} + \dim \tilde{Z} < \dim \tilde{X}$, then $\dim V' < r = \text{rank of } E'$; hence $\dim V'_i < r = \text{rank of } E'_i$. Thus, by Theorem 2 (in the introduction), we have $\chi^{O_{\tilde{X}}}(O_{\tilde{Y}}, O_{\tilde{Z}}) = 0$ in this case. This proof also shows that for any proper closed subscheme $\mathcal{Y}$ of $X'$ such that $\dim \mathcal{Y} + \dim \mathcal{Z} < \dim X'$ and $\mathcal{Y} \cap \mathcal{Z} \subset X'_s$, we have $\chi^{O_{X'}}(O_{\mathcal{Y}}, O_{\mathcal{Z}}) = 0$.

ii) First we prove the following: Let $\mathcal{F}, \mathcal{G}$ be two coherent $O_{\mathbb{P}(\mathbb{P}^{n-1}_K = \mathcal{E})}$ modules such that $\dim \text{Supp}(\mathcal{F}) + \dim \text{Supp}(\mathcal{G}) = n$. Then $\chi^{O_{\mathbb{P}}}(\mathcal{F}, \mathcal{G}) < 0$.

Consider the following exact sequence

$$
0 \rightarrow O_{\tilde{X}}(1) \rightarrow O_{\tilde{X}} \rightarrow O_{\mathcal{E}} \rightarrow 0.
$$

(1)

Tensoring (1) with $\mathcal{F}(t)$, we obtain the following exact sequence

$$
0 \rightarrow \text{Tor}_1^{O_{\tilde{X}}}(O_{\mathcal{E}}, \mathcal{F}(t)) \rightarrow \mathcal{F}(t + 1) \rightarrow \mathcal{F}(t) \rightarrow \mathcal{F}(t) \rightarrow 0.
$$

Hence $\text{Tor}_1^{O_{\tilde{X}}}(O_{\mathcal{E}}, \mathcal{F}(t)) = \mathcal{F}(t + 1)$.

By a spectral sequence argument due to base change from $\tilde{X}$ to $\mathbb{P}$, we have

$$
\chi^{O_{\tilde{X}}}(\mathcal{G}, \mathcal{F}(t)) = \chi^{O_{\mathbb{P}}}(\mathcal{G}, \mathcal{F}(t)) - \chi^{O_{\mathbb{P}}}(\mathcal{G}, \text{Tor}_1^{O_{\tilde{X}}}(\mathcal{E}, \mathcal{F}(t)))
= \chi^{O_{\mathbb{P}}}(\mathcal{G}, \mathcal{F}(t)) - \chi^{O_{\mathbb{P}}}(\mathcal{G}, \mathcal{F}(t + 1)).
$$

Since $\dim \text{Supp}(\mathcal{F}) + \dim \text{Supp}(\mathcal{G}) = n$, by Corollary 1 of Theorem 1 in [D7] we know that $\chi^{O_{\mathbb{P}}}(\mathcal{G}, \mathcal{F}(t))$ is a polynomial function of degree 1 and it is positive for $t \gg 0$.

Hence $\chi^{O_{\tilde{X}}}(\mathcal{G}, \mathcal{F}(t)) < 0$ and is constant for $t \gg 0$. This implies that $\chi^{O_{\tilde{X}}}(\mathcal{G}, \mathcal{F}(t))$ is a constant polynomial of $t \in \mathbb{Z}$ (EGA III, Theorem 2.5.3). Hence $\chi^{O_{\tilde{X}}}(\mathcal{G}, \mathcal{F}) < 0$. 

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Let δ denote imbedding E ↪ X. If Ŷ ∉ E and Ž ⊂ E, then \( \chi^{\mathcal{O}_X}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) = \chi^{\mathcal{O}_Z}(\delta^*(\mathcal{O}_{\tilde{Y}}), \mathcal{O}_{\tilde{Z}}) \) and the required result follows by Corollary 2, Theorem 1 in [D7].

iii) If \( \dim(G(R/P) \otimes G(R/q)) = 0 \), then \( \tilde{Y} \cap \tilde{Z} = \emptyset \). Hence, by i) \( \chi^{\mathcal{O}_X}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) = 0 \). If \( \dim(G(R/P) \otimes G(R/q)) = 1 \), then \( \tilde{Y} \cap \tilde{Z} \) is a finite set of closed points in \( \tilde{X} \), say \( \alpha_1, \ldots, \alpha_t \). Then \( \chi^{\mathcal{O}_X}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) = \sum_{i=1}^{t} \chi^{\mathcal{O}_X, \alpha_i}(\mathcal{O}_{\tilde{Y}, \alpha_i}, \mathcal{O}_{\tilde{Z}, \alpha_i}) \geq 0 \), by non-negativity part of Serre’s conjecture due to Gabber ([G2], [B], [Ho2]).

iv) Let us recall our construction in the paragraph preceding the statement of our theorem. We want to adjust it a bit more to fit our purpose. Let \( Z' \) be a regular alteration of \( \tilde{Z} \). Then \( Z' \) has a closed immersion in \( \mathbb{P}^N_{\tilde{X}} = X' \) for some \( N' > 0 \). Let \( \pi' : X' \to \tilde{X} \) be the projection map and let \( Y' = \pi'^{-1}(\tilde{Y}) \). Since \( Z' \) and \( X' \) are both regular, \( Z' \) is a local complete intersection in \( X' \) and let \( r = \text{codimension of } Z' \text{ in } X' \). Now we repeat this process for \( Y' \). We do it in such a way that in the regular alteration \( Y'' \) of \( Y' \), the inverse image of \( Y' \cap Z' \) is a divisor. This can be achieved by first blowing up \( Y' \) at \( Y' \cap Z' \) and then taking a regular alteration of the blow-up. Again \( Y'' \) has a closed immersion in \( \mathbb{P}^N_{\tilde{X}} = X'' \) for some \( N'' > 0 \); let \( d = \text{codimension of } Y'' \text{ in } X'' \). Let \( \pi' : X'' \to X' \) be the projection and let \( Z'' = \pi''^{-1}(Z') \). Denote by \( I \) and \( J \) the defining ideals of \( Y'' \) and \( Z'' \) in \( X'' \); then \( I \) and \( J \) are local complete intersection ideals in \( \mathcal{O}_{X''} \) of codimension \( d \) and \( r \) respectively. Write \( T = Y'' \cap Z'' \); then \( I + J \) is the defining ideal of \( T \) in \( X'' \) and \( T \subset Y'' \) (the fiber over \( s = [m] \)).

Since \( \pi' \) is flat of relative dimension \( N' \), \( \dim Y' = \dim \tilde{Y} + N' \). By construction \( \dim Z' = \dim Z \); hence \( \dim Y' + \dim Z' = \dim X' \). Since \( \pi' \) is proper, by pulling back a locally free \( \mathcal{O}_{\tilde{X}} \)-resolution of \( \mathcal{O}_{\tilde{Y}} \); we obtain, by projection formula

\[
\chi^{\mathcal{O}_{X'}}(\mathcal{O}_{Y'}, \mathcal{O}_{Z'}) = \chi^{\mathcal{O}_{X}}(\mathcal{O}_{\tilde{Y}}, f_*(\mathcal{O}_{Z'})) + \sum_{i \geq 1} (-1)^i \chi^{\mathcal{O}_{X}}(R^if_*(\mathcal{O}_{Z'}), \mathcal{O}_{\tilde{Y}}).
\]

Since \( \dim (\text{support of } f_*(\mathcal{O}_{Z'})) = \dim (\text{support of } \mathcal{O}_{\tilde{Z}}) \) and for \( i > 0 \) \( \dim (\text{support of } R^if_*(\mathcal{O}_{Z'})) < \dim (\text{support of } \tilde{Z}) \), we conclude by the work done in part i) that \( \chi^{\mathcal{O}_{X'}}(\mathcal{O}_{Y'}, \mathcal{O}_{Z'}) \geq 0 \) according as \( \chi^{\mathcal{O}_{X}}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) \geq 0 \).

Arguing in a similar manner, we conclude that

\[
\chi^{\mathcal{O}_{X''}}(\mathcal{O}_{Y''}, \mathcal{O}_{Z''}) \geq 0 \text{ according as } \chi^{\mathcal{O}_{X}}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) \geq 0.
\]
Now let us get back to the statement of Result 3 at the beginning of this section. In the equicharacteristic case we can do the alteration in such a way that $Y''$ is a non-reduced strict normal crossing divisor in $Y''$. Since $T = Y'' \cap Z'' \subset Y''$ and $T$ is a divisor in $Y''$, due to regularity of $Y''$, $T$ is also a non-reduced strict normal crossing divisor. In the mixed characteristic case if $\alpha$ and $\alpha_1$ are the generators of the maximal ideals in $\nu$ and $\nu_1$ corresponding to $S$ and $S_1$ (Result 3) respectively, then $\alpha = u_1 \alpha_1^h$ in $\nu_1$, $u_1$ a unit. Since $Y''$ is semi-stable over $S_1$, $Y''_{\alpha_1}$ (fiber over the closed point $[\alpha_1] \in S_1$) is a strict normal crossing divisor in $Y''$ and hence $Y''_{\alpha}$ is a non-reduced strict normal crossing divisor in $Y''$. Since $T \subset Y'' \subset Y''$ and $T$, $Y''$ are divisors in the regular scheme $Y''$, $T$ and $Y''$ are non-reduced strict normal crossing divisors in this case too.

Let $E = \text{Spec}(\text{Sym}_{\O_{Y''}}(I/I^2))$—the normal bundle to the imbedding $Y'' \hookrightarrow X''$ and let $V = \text{Spec}(\bigoplus I_t \O_{Z''}/I^{t+1} \O_{Z''})$ be the normal cone to $Y'' \cap Z''$ in $Z''$. $E$ is a vector-bundle of rank $d$ on $Y''$. By repeating the argument of Gabber (as in the proof of part i) we have

$$\chi^{\O_{X''}}(\O_{Y''}, \O_{Z''}) = \chi^E(\O_{Y''}, \O_V).$$

Claim 1. $V$ is a locally trivial bundle of rank $(d + 1 - r)$ over $T$.

The proof of this claim follows from local considerations. Let $A$ be the co-ordinate ring of a sufficiently small neighborhood in $X''$ of a closed point $Q$ in $T$ and let $I$ and $J$ represent $\mathcal{I}$ and $\mathcal{J}$ in $A$. Then $T = \text{Spec}(A/(I + J))$, $Y'' = \text{Spec}(A/I)$, $Z'' = \text{Spec}(A/J)$. Since $T$ is an effective divisor in $Y''$, $(I + J)/I$ is generated by a non-zero-divisor in $A/I$. Since $I$ and $J$ are complete intersections in $A$, can write $I = (b_1, \ldots, b_d)$ and $J = (a_1, \ldots, a_r)$. Since $A_Q$, $(A/I)_Q$ and $(A/J)_Q$ are regular, can assume that each of $\{b_1, \ldots, b_d\}$ and $\{a_1, \ldots, a_r\}$ form a part of a regular system of parameters for $A_Q$. Since $(I + J)/I$ is generated by a non-zero-divisor in $A/I$, $\{a_1, \ldots, a_r\}$ can be so chosen that $a_i = b_i$ for $1 \leq i \leq r - 1$ and $\text{im}(a_r) = \text{im}(J)$ in $A/I$. Thus $a_1, \ldots, a_r, b_r, \ldots, b_d$ form an $A$-sequence (may have to shrink Spec $A$ more). This implies that in $A/J$, $(I + J)/J = \text{im}(I)$ form an $A/J$ sequence: i.e., $\text{im}(b_r), \ldots, \text{im}(b_d)$ form an $A/J$ sequence of length $(d + 1 - r)$. Hence $V$ is a locally trivial bundle of rank $(d + 1 - r)$ over $T$.

Let $X''$, $E_s$, $Y''$ denote the fibers of $X''$, $E$, $Y''$ respectively over $s$. Then $X''_s = \mathbb{P}^{n-1}_K \times \ldots \times \mathbb{P}^{n-1}_K$.
Since \( T \subset Y_s'' \) and both are effective cartier divisors in \( Y'' \), irreducible components of \( T \) are also irreducible components of \( Y_s'' \). Write \( W = Y_s'' \). Let \( W_1, \ldots, W_h, V_1, \ldots, V_e \) and \( E_1, \ldots, E_h \) denote the components of \( W, V \) and \( E_s \) respectively (\( E_i = W_i \times Y_s' \)). Let \( m_i, n_i \) denote the geometric multiplicities of \( W_i, V_i \) respectively in \( W \) and \( V \).

By claim 1, each component of \( V \) is contained in a unique component of \( E_s \). Assume \( V_i \subset E_i, 1 \leq i \leq e \). Then we have

\[
\chi^{\mathcal{O}_E}(\mathcal{O}_V, \mathcal{O}_{Y''}) = \sum_{i=1}^{e} n_i \chi^{\mathcal{O}_{E_i}, \mathcal{O}_{W_i}}. 
\] (3)

Let \( i = 1 \). We would like to prove the following.

**Claim 2** There exists a vector bundle \( E'_1 \) on \( W_1 \) of rank \( d \) generated by global sections and a map \( \phi : E'_1 \to E_1 \) such that the following holds:

A1) if \( m_1 = 1 \), then \( \phi \) is dominating, \( \dim \phi^{-1}(V_1) \leq \text{rank } E'_1 \) and \( \text{Tor}^{E_1}_i(\mathcal{O}_{V_1}, \mathcal{O}_{E'_1}) = 0 \) for \( i > 0 \) and

A2) if \( m_1 > 1 \), then \( \text{im} \phi \) is a vector bundle of rank \((d - 1)\).

**Part A1)** of claim 2 implies part iv) of our theorem:

By the Corollary stated in the introduction, Part A1) of claim 2 implies that \( \chi^{\mathcal{O}_{E_1}}(\mathcal{O}_{V_1}, \mathcal{O}_{W_1}) \geq 0 \) if \( m_1 = 1 \). The same conclusion holds if \( m_i = 1 \) for every \( i \). Hence, by (3), \( \chi^{\mathcal{O}_{E}}(\mathcal{O}_{V}, \mathcal{O}_{Y''}) \geq 0 \) and proof of part iv) of our theorem is complete.

**Proof of claim 2.** We have \( W_1 \subset W = Y_s'' \subset Y'' \subset X'' \). Recall that \( E = \text{Spec} (\text{Sym}_{\mathcal{O}_{Y''}} (\mathcal{I}/\mathcal{I}^2)) \) is the normal bundle corresponding to the regular imbedding \( Y'' \hookrightarrow X'' \). Let \( \mathcal{I}_1 \) be the ideal of definition of \( W_1 \) in \( X'' \). Then \( \mathcal{I} + m\mathcal{O}_{X''} \subset \mathcal{I}_1 \), \( E_1 = \text{Spec} (\text{Sym}_{\mathcal{O}_{W_1}} (\mathcal{I}/\mathcal{I}_1 \mathcal{I})) \) and \( \mathcal{I}_1 \) is the ideal of definition of \( W_1 \) in \( X''_s \). Let \( E'_1 = \text{Spec} (\text{Sym}_{\mathcal{O}_{W_1}} (\mathcal{I}_1/\mathcal{I}_1^2)) \) denote the normal bundle corresponding to the regular imbedding \( W_1 \in X''_s \). Since \( T_{X''_s}/K \) is generated by global sections and \( W_1 \) is a smooth subvariety of \( X''_s \) (\( Y_s'' \) being a non-reduced strict normal crossing divisor in \( Y'' \)), \( E'_1 \) is generated by global sections. The map \( \phi : E'_1 \to E_1 \) in our claim is induced by the natural map

\[
\psi : \mathcal{I}/\mathcal{I}_1 \mathcal{I} \to \mathcal{I}_1/\mathcal{I}_1^2 = \mathcal{I}_1/(\mathcal{I}_1^2 + m\mathcal{O}_{X''}). \] (4)
To show that $\phi$ possesses the properties mentioned in our claim we need to concentrate on the local picture.

**Local Picture** (in a small affine neighborhood in $X''$ of a closed point in $W_1$). Let $x = 0$ represent a local equation of $E(E_{\tilde{Z}}, E_{\tilde{Y}})$ in an affine open net in $\tilde{X}$ ($\tilde{Y}, \tilde{Z}$) (actually $x = x_i$ where $\{x_1, \ldots, x_n\}$ is a regular system of parameters of $m$ and the corresponding affine open set is $\text{Spec}(R[x_1/x_i, \ldots, x_n/x_i])$). Let $A$ be a regular domain such that $X'' = \text{Spec} A$ and $X''_s = \text{Spec}(A/xA)$. Let $\Gamma(\text{Spec} A, \mathcal{I}) = I$, and $\Gamma(\text{Spec} A, \mathcal{J}) = J$. Note that $A/xA$, $A/I$ and $A/J$ are all regular (shrink $A$ if necessary). Write $\overline{A} = A/xA$, $\tilde{A} = A/I$ and $\tilde{x} = \text{im}(x)$ in $\tilde{A}$. Then $Y'' = \text{Spec} \tilde{A}$, $X''_s = \text{Spec} \overline{A}$, $Y''_s = \text{Spec}(\overline{A}/\overline{T})$ and $Z'' = \text{Spec}(A/J)$. $I$ and $J$ are both generated by regular sequence $\{b_1, \ldots, b_d\}$ and $\{a_1, \ldots, a_r\}$ respectively such that $a_i = b_i$ for $i < r$ and $\{a_1, \ldots, a_r, b_r, \ldots, b_d\}$ form an $A$-sequence (proof of claim 1). Moreover $\text{im}(I)$ in $\overline{A} = \{\overline{b}_r, \ldots, \overline{b}_d\}$ form an $\overline{A}$-sequence. Since $T$, $Y''_s$ are non-reduced strict normal crossing divisors in $Y''$ and $T \subset Y''_s$, there exists $t_1, \ldots, t_j$ in $A$ such that

$$\tilde{x} = u \tilde{t}_1^{m_1} \cdots \tilde{t}_j^{m_j}, \ j \leq h \tag{5}$$

in $\tilde{A}$, where $u$ is a unit, $\tilde{t}_i = \text{im} t_i$, $1 \leq i \leq c$, $c \leq j$ in $\tilde{A}$ corresponds to the component $W_i$ of both $T$ and $Y''_s$; $\overline{A}/(\overline{t}_1, \ldots, \overline{t}_i)$, $i \leq j$, is a regular ring of codimension $i$ in $\overline{A}$ ($Y''_s$ being a non-reduced strict normal crossing divisor). Then $W_1 = \text{Spec}(\tilde{A}/t_1 \tilde{A})$, $E = \text{Spec}(\text{Sym}_{\tilde{A}}(I/I^2))$, $E_s = \text{Spec}(\text{Sym}_{\overline{A}/x\overline{A}}((I^2 + xI)/I))$, $E_1 = \text{Spec}(\text{Sym}_{\tilde{A}/t_1\tilde{A}}((I^2 + xI)/I))$ and $E'_1 = \text{Spec}(\text{Sym}_{\overline{A}/t_1\overline{A}}((I, t_1)/(I, t_1)^2 + xA))$. Since $A/(I, t_1) (= \overline{A}/(\overline{I}, \overline{t}_1))$ is regular of codimension $d + 1(d)$ in $A(\overline{A})$ and $A, \overline{A}$ are regular, $x$ is a minimal generator of $(I, t_1)$ and its image is a part of a basis of $(I, t_1)/(I, t_1)^2$. Write $a = ut_2^{m_2} \cdots t_j^{m_j}$; then, by (5), $x - at_1^{m_1} \in I$. Hence $x - at_1^{m_1} = \sum \lambda_i b_i$ where one of the $\lambda_i$ is a unit (may have to shrink $A$ to achieve this). (The referee pointed out that there is a possibility that none of the $\lambda_i$s is a unit in the case where $a$ is a unit and $m_1 = 1$. In this case, $\psi$ and $\phi$ are isomorphisms.) Thus without any loss of generality we can assume

$$at_1^{m_1} - x = b_\beta \text{ for some } \beta, \ 1 \leq \beta \leq d. \tag{6}$$

We fix this $\beta$. 

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The map $\phi : E_1^\prime \to E_1$ is induced by

$$
\psi : I/(I^2 + t_1I) \to (I, t_1)/( (I, t_1)^2 + xA)
$$

(7)

where $\psi(\text{im } b_i) = \text{im}(b_i), i \neq \beta$ and $\psi(\text{im } b_\beta) = \text{im}(at_1^{m_1})$.

**Proof of Part A.** Assume $m_1 = 1$.

Then in the above map $\psi(\text{im } b_\beta) = \text{im}(at_1)$. So, locally $\varphi$ is defined by $\tilde{\psi}$

$$
\tilde{\psi} : A/(I, t_1) [X_1, \ldots, X_d] \to A/(I, t_1) [X_1, \ldots, X_{d-1}, Z_d]
$$

where $\tilde{\psi}(X_i) = X_i, 1 \leq i \leq d - 1$ and $\tilde{\psi}(X_d) = aZ_d$.

Write $B = A/(I, t_1)[X_1, \ldots, X_{d-1}]$. Then

$$
\tilde{\psi} : B[X_d] \to B[Z_d] \text{ is given by } \tilde{\psi}|_B = \text{Id}_B \text{ and } \tilde{\psi}(X_d) = aZ_d.
$$

(8)

Let $P_1$ be the defining prime ideal for $V_1$ in $B[X_d]$. Since $V_1 \subset E_1$ only, $a \notin P_1$. Since $B[X_d]_a \cong B[Z_d]_a$, $\text{Tor}_i^{B[X_d]}(B[X_d]/P_1, B[Z_d])$ are annihilated by $a^t$ for some $t > 0$ and for all $i > 0$. Since the above Tors are taken over $B[X_d]$, we will drop this term from notations of Tors for convenience of typing. We have two cases to consider.

**Case 1.** $X_d \in P_1$. In this case $P_1 = (P_0, X_d), P_0$ a prime ideal of $B$.

Consider the exact sequence

$$
0 \to B[X_d]/P_0 \xrightarrow{X_d} B[X_d]/P_0 \to B[X_d]/P_1 \to 0.
$$

Since $\text{Tor}_i(B[X_d]/P_0, B[Z_d]) = 0$ for $i > 0$. We have $\text{Tor}_i(B[X_d]/P_1, B[Z_d]) = 0$ for $i > 1$ and we obtain the following exact sequence

$$
0 \to \text{Tor}_1(B[X_d]/P_1, B[Z_d]) \to B[Z_d]/P_0 \xrightarrow{aZ_d} B[Z_d]/P_0 \to B[Z_d]/(P_0, aZ_d) \to 0
$$

by tensoring the above sequence with $B[Z_d]$. Since $a \notin P_1$ and $Z_d \notin P_0$, $\text{Tor}_1(B[X_d]/P_1, B[Z_d]) = 0$ and $\text{dim}(B[X_d]/P_1 \otimes B[Z_d]) = \text{dim}(B[Z_d]/(P_0, aZ_d)) = \text{dim} B[X_d]/P_1$.

**Case 2.** $X_d \notin P_1$. In this case $X_d$ is a non-zero-divisor on $B[X_d]/P_1$; we write $(P_1, X_d) = (P_0, X_d)$ where $P_0$ is an ideal of $B$ and $\text{dim} B/P_0 = \text{dim} B[X_d]/(P_1, X_d) = \text{dim} B[X_d]/P_1 - 1$.
Consider the exact sequence

$$0 \rightarrow B[X_d]/P_1 \xrightarrow{X_d} B[Z_d]/P_1 \rightarrow B[X_d]/(P_0, X_d) \rightarrow 0.$$ 

As in case 1, we have $\text{Tor}_i(B[X_d]/(P_0, X_d), B[Z_d]) = 0$ for $i \geq 2$. Hence $\text{Tor}_i(B[X_d]/P_1, B[Z_d])$ is an isomorphism for $i \geq 2$ and is an injection for $i = 1$. Since $a^t$ annihilates these Tors, $\text{Tor}_i(B[X_d]/P_1, B[Z_d]) = 0$ for $i \geq 1$. Since $B[X_d]_{a} \cong B[Z_d]_{a}$ and $\dim B/P_0 = \dim B[X_d]/P_1 - 1$, by tensoring the above equation with $B[Z_d]$, we have $\dim(B[X_d]/P_1 \otimes B[Z_d]) = \dim B[X_d]/P_1$.

Hence proof of Part $A_1$ of claim 2 is complete and so the proof of iv) follows.

**Proof of Part $A_2$**. Assume $m_1 > 1$. Then $\psi(\text{im} b_\beta) = 0$ in (7) and $\tilde{\psi} : B[X_d] \rightarrow B[Z_d]$ is given by $\tilde{\psi}|_B = I_d$ and $\tilde{\psi}(X_d) = 0$ in (8) where $B = A/(I, t_1) [X_1, \ldots, X_{d-1}]$. Hence $B[X_d]/\ker \tilde{\psi} = B$. Thus part $A_2$ of claim 2 is proved.

Let $F_1$ denote the image bundle $\text{im}(\phi)$. Then $F_1$ is subbundle of $E_1$ of rank $(d - 1)$.

**Proof of v)**. We have two cases to consider.

**Case 1.** $X_d \notin P_1 \text{ i.e., } V_1 \text{ is not contained in } F_1$. Then $X_d$ is a non-zero-divisor on $B[X_d]/P_1$. We write $(P_1, X_d) = (P_0, X_d)$ where $P_0$ is an ideal of $B$ and $\dim B/P_0 = \dim B[X_d]/(P_1, X_d) = \dim B[X_d]/P_1 - 1$. Consider the exact sequence

$$0 \rightarrow B[X_d]/P_1 \xrightarrow{X_d} B[Z_d]/P_1 \rightarrow B[X_d]/(P_0, X_d) \rightarrow 0.$$ 

As in case 1 of part $A_1$ in claim 2, $\text{Tor}_i(B[X_d]/(P_0, X_d), B[Z_d]) = 0$ for $i \geq 2$. Hence $\text{Tor}_i(B[X_d]/P_1, B[Z_d]) \xrightarrow{\text{im} X_d = 0} \text{Tor}_i(B[X_d]/P_1, B[Z_d])$ is injective for $i \geq 1$; consequently all the Tors vanish for $i \geq 1$. Moreover, $B[X_d]/P_1 \otimes B[Z_d] = B[X_d]/(P_0, X_d) \otimes B[Z_d] = B[Z_d]/P_0$; hence $\dim B[X_d]/P_1 \otimes B[Z_d] = \dim B[X_d]/P_1$. Thus, by the Corollary stated in the introduction, $\chi^{O_{E_1}}(O_{V_1}, O_{W_1}) = \chi^{O_{E_1'}}(O_{\phi^{-1}(V_1)}, O_{W_1}) \geq 0$.

**Case 2.** $X_d \in P_1 \text{ i.e., } V_1 \text{ is contained in } F_1$. In this case we will show that $\chi^{O_{E_1}}(O_{V_1}, O_{W_1}) \leq 0$.

We need to take a deeper look at the situation in part $A_2$ of claim 2. In this case the collection of $\{\text{im} b_\beta\}$ in $I/(I^2 + t_1 I)$ defines an effective cartier divisor $D$ on $E_1$ whose support is the locally trivial bundle $F_1$ of rank $(d - 1)$ on $W_1$. 18
Claim 3. Let $\mathcal{L}_1 = \text{Ker } \psi ((4), (7))$. Then $\mathcal{L}_1 = f^*(\mathcal{O}_F(1))$ where $f : W_1 \rightarrow \mathbb{P}^{n-1} = \mathbb{P}$ is obtained by restricting $\pi' \circ \pi''$ to $W_1$.

Proof of the claim. The proof follows from local considerations. Suppose that $m$, the maximal ideal of $R$ is generated by $x_1, \ldots, x_n$. Let $D_+(x_i)$ denote the standard affine open sets of $\tilde{X}(\tilde{Y})$ and let $D_+(\mathfrak{p}_i)$ denote the standard affine open sets of $\mathcal{E} = \mathbb{P}(\mathcal{E}_Y)$. Let $Q$ be a closed point in $f^{-1}(D_+(\mathfrak{p}_1) \cap D_+(\mathfrak{p}_2))$. As in (5), in a small affine neighborhood of $Q$ in $\mathbb{P}^{n-1}$, we have

$$\tilde{x}_i = \tilde{\gamma}_i \tilde{t}^{n_1}_1 \cdots \tilde{t}^{n_\ell}_\ell \text{ and } \tilde{x}_j = \tilde{\gamma}_j \tilde{s}^{r_1}_1 \cdots \tilde{s}^{r_\ell}_\ell, \quad \ell \leq d, \quad h \leq d,$$

where $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$ are units. \hfill (9)

In $\Gamma(D_+(x_i, x_j), \mathcal{O}_X)$, $x_j = (x_j/x_i)x_i$ where $x_j/x_i$ is a unit. Similar equations hold in $\Gamma(D_+(\mathfrak{p}_j, \mathfrak{p}_j), \mathcal{O}_\mathfrak{p})$. Thus from (9) we have

$$\tilde{\gamma}_i \tilde{t}^{n_1}_1 \cdots \tilde{t}^{n_\ell}_\ell = \tilde{\gamma}_j (\tilde{x}_i/\tilde{x}_j) \tilde{s}^{r_1}_1 \cdots \tilde{s}^{r_\ell}_\ell.$$

Since $\mathbb{P}^{n-1}$ is regular, this implies, $\ell = h$, $n_\mu = r_\mu$, $1 \leq \mu \leq \ell$ and there exists units $\alpha_\mu$ such that $t_\mu - \alpha_\mu s_\mu \in I$ for $1 \leq \mu \leq \ell$. Since $x_i = \gamma_i t^{n_1}_1 \cdots t^{n_\ell}_\ell + b_\beta_i$ and $x_j = \gamma_j s^{r_1}_1 \cdots s^{r_\ell}_\ell + b_\beta_j$ \hfill (7), we have $(x_j/x_i)b_\beta_i - b_\beta_j \in (I^2, t_1I)$. Hence $(\mathfrak{p}_j/\mathfrak{p}_i)\text{im } b_\beta_i = \text{im } b_\beta_j$ in $I/(I^2, t_1I)$.

Thus $\mathcal{L}_1 = f^*(\mathcal{O}_F(1))$.

Since $E'_1$ is generated by global sections, it follows from claim 1 that $F_1$ is also generated by global sections.

Recall that $V_1 \subset F_1$.

Let $i$ denote the natural inclusions for both $F_1 \hookrightarrow E_1$ and $\mathbb{P}(F_1 \oplus 1) \hookrightarrow \mathbb{P}(E_1 \oplus 1)$. Denote by $\xi_1$ the universal quotient bundle of rank $r$ of $q_1^*(E_Y^\vee \oplus 1)$, where $q_1 : \mathbb{P}(E_1 \oplus 1) \rightarrow W_1$ is the projection map. Let $q = q_1|_{\mathbb{P}(F_1 \oplus 1)}$ and $\xi = q_1^*(E_Y^\vee \oplus 1)$. This set-up leads to the following exact sequence

$$0 \rightarrow \xi \rightarrow i^*(\xi_1) \rightarrow q^*(\mathcal{L}_1^\vee) \rightarrow 0. \quad \text{(10)}$$
We have

\[
\chi^{\mathcal{O}_{E_1}}(\mathcal{O}_{V_1}, \mathcal{O}_{W_1}) = \int c_r(\xi_1)([\mathcal{N}_1]) \quad \text{(part iv, Theorem 2 in the introduction)}
\]

\[
= \int c_1(q^*(L_{i_1}^\vee))c_{r-1}(\xi) \cap [\mathcal{N}_1] \quad \text{(exact sequence (10))}
\]

\[
= \int c_1(L_{i_1}^\vee) \cap \delta, \quad \text{where } \delta = q_1^*[c_{r-1}(\xi) \cap \mathcal{N}_1].
\]

Since \( F_1 \) is generated by global sections, by result 3 of this section, \( \delta = \omega_1^* (V_1) \in A_{E_1}^\infty(W_1) \) where \( \omega_1 : W_1 \to E_1 \) is the 0-section. Since \( f_*[c_1(L_{i_1}^\vee) \cap \delta] = c_1(\mathcal{O}_{\mathcal{P}}(-1)) \cap f_*\delta \), we have

\[ \int c_1(L_{i_1}^\vee) \cap \delta \leq 0. \]

Thus \( \chi^{\mathcal{O}_{E_1}}(\mathcal{O}_{V_1}, \mathcal{O}_{W_1}) \leq 0 \) and the proof of our theorem is complete.

**Corollary.** Let \( R \) be a regular local ring and let \( M, N \) be two finitely generated \( R \)-modules such that \( \ell(M \otimes_R N) < \infty \). We have the following:

i) if \( \dim M + \dim N < \dim R \), then \( \chi(M, N) = 0 \).

ii) Suppose \( \dim M + \dim N = \dim R \).

a) if \( \dim(G(M) \otimes G(N)) \leq 1 \), then \( \chi(M, N) \geq e_m(M)e_m(N) \) and

b) \( M = R/P, N = R/q \). If \( m_{\sigma(i)} = 1 \) for \( 1 \leq i \leq e \) (notations as in the statement of the above theorem), then \( \chi(R/P, R/q) \geq e_m(R/P)e_m(R/q) \).

Proof of this Corollary follows from Theorem 1 in the introduction and the above theorem. Tennyson [T] proved the assertion in subpart a) of part ii) of this corollary when \( \dim(G(M) \otimes G(N)) = 0 \).
References


Department of Mathematics
University of Illinois
1409 West Green Street
Urbana, IL 61801
U.S.A.
e-mail: dutta@math.uiuc.edu