MATH 416
Practice midterm 2

(1) Let \( A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \\ 6 & 3 \end{pmatrix} \) and let \( L_A : \mathbb{R}^2 \to \mathbb{R}^3 \) be the corresponding linear transformation.

(a) Compute \( L_A(1, 2) \).

(b) Find a basis for \( N(L_A) \).

(c) Find a basis for \( R(L_A) \).

(2) Let \( \mathbb{R}^\infty \) be the vector space of sequences of real numbers. Define \( T : \mathbb{R}^\infty \to \mathbb{R}^\infty \) by \( T((a_1, a_2, a_3, \ldots)) = (a_2, a_3, \ldots) \). Prove that \( T \) is linear. Is \( T \) one-to-one? Is \( T \) onto?

(3) (a) Let \( V \) and \( W \) be vector spaces. Define what it means for \( T : V \to W \) to be an isomorphism.

(b) Determine whether the following linear transformation is an isomorphism.

\[
T : P_3(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R}) \text{ defined by } T(a + bx + cx^2 + dx^3) = \begin{pmatrix} a-b & b-c \\ c-d & d-a \end{pmatrix}
\]

(4) (a) State the dimension theorem.

(b) Define \( T : P_3(\mathbb{R}) \to P_3(\mathbb{R}) \) by \( T(p(x)) = p'(x) \). Find \( \text{nullity}(T) \) and \( \text{rank}(T) \).

(5) Define \( T : P_2(\mathbb{R}) \to P_2(\mathbb{R}) \) by \( T(p(x)) = p(x) + p'(x) \). Let \( \beta = \{1, x, x^2\} \) and \( \gamma = \{x^2, x^2 + 2x, x^2 + 4x + 2\} \).

(a) Compute \( [T]_{\beta} \) and the change of coordinate matrices \( [I_{P_2(\mathbb{R})}]_{\gamma}^{\beta} \) and \( [I_{P_2(\mathbb{R})}]_{\beta}^{\gamma} \).

(b) Use part (a) to compute \( [T]_{\gamma} \).
(6) Let $A \in M_{m\times n}(\mathbb{R})$.

(a) Define the rank of $A$.

(b) Prove that $\text{rank}(A) = \text{rank}(A^t)$. (You may use any theorems/results proven or given in class.)

(7) Let $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$.

(a) Compute $\det(A)$ using cofactor expansion along any row of $A$.

(b) Row reduce $A$ and use the resulting matrix in row echelon form to compute $\det(A)$. 