Optimization

1. Suppose $y = \frac{x^2}{x^2 + 1}$ for $x > 0$. Determine the $x$-coordinate for the point on this curve which is closest to the origin.

$$
\text{minimize } D = \sqrt{x^2 + y^2} \Rightarrow \text{minimize } S = D^2 = x^2 + y^2
$$

$$
S = x^2 + \left(\frac{x^2}{x^2 + 1}\right)^2 = x^2 + \frac{2x^2}{x^2 + 1}
$$

$$
S' = 2x - \frac{2x^2}{(x^2 + 1)^2} = 2x - \frac{2x}{x^2 + 1} = 2\frac{x^3 - 75}{x^2 + 1}
$$

$x = 8\sqrt{75}$

2. A farmer wants to fence in a rectangular plot of land adjacent to the north wall of his barn. No fencing is needed along the barn, and the fencing along the west side of the plot is shared with a neighbor who will split the cost of that portion of the fence. If the fencing costs $20 per foot to install and the farmer is not willing to spend more than $5000, find the dimensions for the plot that would enclose the most area.

$$
\text{Cost} = C = \frac{1}{2}(20x) + 20y + 20x = 20y + 30x
$$

Maximize area - don't care about cost, so use $5000$.

$5000 = 20y + 30x \Rightarrow y = 250 - \frac{3}{2}x$

$$
\text{Area} = A = xy = x(250 - \frac{3}{2}x) = 250x - \frac{3}{2}x^2
$$

A' = 250 - 3x

A' = 0 when $x = \frac{250}{3}$.

A'' = -3 < 0, so by second deriv. test, $\frac{250}{3}$ is maximum.

$$
\text{max at } \left(\frac{250}{3}, 125\right)
$$
Definite Integrals as Limits of Riemann Sums

1. Fill in the missing information to show that the given definite integral can be expressed as the limit of a right Riemann sum. The only variables appearing in your limit should be \( n \) and \( k \). You do not need to evaluate this limit.

\[
\int_{3}^{9} (x^2 + 5)^4 \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} \left[ \left( \left( 3 + \frac{k\Delta x}{n} \right)^2 + 5 \right)^4 \cdot \frac{6}{n} \right] \quad \Delta x = \frac{6}{n} \\
\chi_k = 3 + k\Delta x
\]

2. Express the limit as a definite integral. *Hint: Consider* \( f(x) = x^3 \).

\[
\lim_{n \to \infty} \sum_{i=1}^{n} 3\left( \frac{3i^3}{n^3} \right) \frac{1}{n} = \lim_{n \to \infty} \sum_{i=1}^{n} f(\chi_i) \Delta x = \int_{1}^{3} 3x^3 \, dx
\]

\[
\Rightarrow f(\chi_i) = 3x^3 \quad \Delta x = \frac{1}{n}
\]

3. Evaluate the Riemann sum for \( f(x) = x - 1 \), \(-6 \leq x \leq 4\) with five subintervals, taking the sample points to be left endpoints. Is this an overestimate or underestimate?

\[
\Delta x = \frac{b-a}{n} = \frac{4-(-6)}{5} = 2
\]

\[
R_5 = \sum_{i=1}^{5} f(\chi_i) \Delta x
\]

\[
= \Delta x \left[ f(-6) + f(-4) + f(-2) + f(0) + f(2) \right]
\]

\[
= 2 \left[ (-7) + (-5) + (-3) + (-1) + (1) \right]
\]

\[
= 2 [-15] = -30
\]

underestimate.
1. A spherical balloon is being inflated so that its diameter is increasing at a constant rate of 8 cm/min. How quickly is the volume of the balloon increasing when the diameter is 20 cm?

\[
D = \text{diameter, } r = \text{radius, } V = \text{volume} \\
\frac{dD}{dt} = 8 \\
V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \left(\frac{D}{2}\right)^3 = \frac{\pi}{6}D^3 \\
\frac{dV}{dt} = \frac{d}{dt} \left(\frac{\pi}{6}D^3\right) = \frac{\pi}{2}D^2 \frac{dD}{dt} = \frac{\pi}{2}D^2 \cdot 8 = 4\pi D^2
\]

@ D = 20 cm, \(\frac{dV}{dt} = 4\pi (20)^2 = \boxed{1600\pi \text{ cm}^3/\text{min}}\)

2. As Emily walks away from a 510 cm lamppost, the tip of her shadow moves 50% faster than she does. What is Emily’s height?

\[
given: \quad \frac{dy}{dt} = \frac{dx}{dt} + 1.5 \frac{dx}{dt} = 1.5 \frac{dx}{dt}
\]

\(y\) is tip of shadow position  
\(x\) is Emily’s position  
want H.

Similar triangles \(\Rightarrow \frac{510}{H} = \frac{y}{y-x}\)

\[H \frac{dy}{dt} = 510 \frac{dy}{dt} - 510 \frac{dx}{dt}\]

\[H (1.5 \frac{d^2 x}{dt}) = 510 (1.5 \frac{dx}{dt}) - 510 \frac{dx}{dt}\]

\[1.5H = 510 (1.5 - 1)\]

\[H = \frac{510 (1.5 - 1)}{1.5} = \frac{510 \cdot 1.5}{1.5} = \boxed{170\text{ cm}}\]
Exponential Growth and Decay

1. A curve passes through the point \((4, e^7)\) and has the property that for each point on the curve, the slope of the curve is equal to half of the \(y\)-coordinate. What is the equation of the curve?

\[
\frac{dy}{dx} = \frac{1}{2} y \Rightarrow y = Ce^{\frac{1}{2}x} \tag{4, e^7}
\]

\[
e^7 = Ce^{\frac{1}{2} \cdot 4} \Rightarrow C = \frac{e^7}{e^2} = e^5
\]

\[
\Rightarrow y = e^5e^{\frac{1}{2}x} = e^{\frac{1}{2}x + 5}
\]

2. When a cold drink is taken from a refrigerator, its temperature is \(5^\circ C\). After 25 minutes, the temperature has increased to \(10^\circ C\). Assuming the relative rate at which the drink is cooling is constant,

\[y = T - 20\]

(a) What is the temperature of the drink after 50 minutes?

\[
\frac{dy}{dt} = ky \Rightarrow y = Ce^{kt} \\
C = y(0) = T(0) - 20 = 5 - 20 = -15
\]

\[
y(25) = -15 e^{kt} \\
y(25) = T(25) - 20 = 10 - 20 = -10
\]

\[
e^{25k} = \frac{10}{-15} = \frac{2}{3}
\]

\[
\Rightarrow k = \ln \left(\frac{2}{3}\right)
\]

\[
y(t) = -15 e^{\ln \left(\frac{2}{3}\right)t/25}
\]

\[
y(25) = -15 \left(\frac{2}{3}\right)^{t/25}
\]

(b) When will its temperature be \(15^\circ C\)?

\[
15 = 20 + y(t)
\]

\[
5 = -15 \left(\frac{2}{3}\right)^{t/25}
\]

\[
\frac{1}{3} = \left(\frac{2}{3}\right)^{t/25}
\]

\[
\ln \left(\frac{1}{3}\right) = \ln \left(\frac{2}{3}\right) \Rightarrow t = \frac{25 \ln \left(\frac{1}{3}\right)}{\ln \left(\frac{2}{3}\right)}
\]
Properties of Summation Notation

Evaluate the following sums and limits, i.e. your answers should not include any variables (do not worry about simplifying further).

1. $\sum_{i=1}^{3} 3i^2 + 4i + 5$
   
   \[
   = \left(3(1)^2 + 4 + 5\right) + \left(3(2)^2 + 4 \cdot 2 + 5\right) + \left(3(3)^2 + 4 \cdot 3 + 5\right)
   \]

2. $\sum_{i=1}^{150} 3i^2 + 4i + 5$
   
   \[
   = 3 \sum_{c=1}^{150} i^2 + 4 \sum_{c=1}^{150} i + 5 \sum_{c=1}^{150} 1
   \]
   
   \[
   = 3 \frac{150(151)(2 \cdot 150 + 1)}{6} + 4 \frac{150(150+1)}{2} + 5 \cdot 150
   \]

3. $\lim_{n \to \infty} \sum_{i=1}^{n} \left[\left(\frac{3}{n}\right)^3 - 6\left(\frac{3}{n}\right)\right] \frac{3}{n}$
   
   \[
   = \lim_{n \to \infty} \frac{3}{n} \sum_{c=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n}i\right]
   \]
   
   \[
   = \lim_{n \to \infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]
   \]
   
   \[
   = \lim_{n \to \infty} \left[\frac{81}{n^4} \left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2} \left(\frac{n(n+1)}{2}\right)\right]
   \]
   
   \[
   = \lim_{n \to \infty} \left[\frac{81}{n^4} \left(\left(1 + \frac{1}{n}\right)^2 - 2 \cdot 7 \left(1 + \frac{1}{n}\right)\right)\right]
   \]
   
   \[
   = \frac{81}{4} - 27
   \]
Limits

Evaluate the following limits without the use of derivatives, or explain why the limit does not exist.

1. \[ \lim_{x \to 0} (12 + 6\ln(1 + 3x^2)) = 12 + 6 \ln (1) = 12 + 6(0) = 12 \]

2. \[ \lim_{x \to 3} \frac{x^2 + 5x - 24}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+8)}{x-3} = \lim_{x \to 3} (x+8) = 11 \]

3. \[ \lim_{x \to 6} \frac{2x + 12}{|x+6|} = \begin{cases} x+6 & x > -6 \\ -(x+6) & x < -6 \end{cases} \]

   One-sided limits:

   \[ \lim_{x \to 6^+} \frac{2x+12}{|x+6|} = \lim_{x \to 6^+} \frac{2(x+6)}{x+6} = 2 \]

   \[ \lim_{x \to 6^-} \frac{2x+12}{|x+6|} = \lim_{x \to 6^-} \frac{2(x+6)}{-(x+6)} = \lim_{x \to 6^-} -\frac{2(x+6)}{x+6} = -2 \]

   Since limits are not the same, \( \lim \) d.n.e.

4. \[ \lim_{x \to 0} \left( \frac{1}{x\sqrt{1+x}} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{1 - \sqrt{1+x}}{x\sqrt{1+x}} \cdot \frac{(1 + \sqrt{1+x})}{(1 + \sqrt{1+x})} \]

   \[ = \lim_{x \to 0} \frac{-x}{x\sqrt{1+x} (1 + \sqrt{1+x})} = \lim_{x \to 0} -\frac{1}{\sqrt{1+x} (1 + \sqrt{1+x})} \]

   \[ = -\frac{1}{2} \]
Asymptotes

1. Find all horizontal asymptotes on the graph of

\[ f(x) = \frac{24 + 3e^{12x}}{8e^{3x} + 6} \]

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{24 + 3e^{12x}}{8e^{3x} + 6} \cdot \frac{\frac{1}{e^{3x}}}{\frac{1}{e^{3x}}} = \lim_{x \to \infty} \frac{24}{8 + \frac{6}{e^{3x}}} \to 8 = 8
\]

\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{24 + 3e^{12x}}{8e^{3x} + 6} = \frac{24 + 3 \cdot 0}{8 \cdot 0 + 6} = \frac{24}{6} = 4
\]

So there is a horizontal asymptote at \(y = 4\).

2. Find all (horizontal and vertical) asymptotes on the graph of

\[ f(x) = \frac{1 + x^4}{x^2 - x^4} \]

\[
\lim_{x \to \pm\infty} \frac{1 + x^4}{x^2 - x^4} = \lim_{x \to \pm\infty} \frac{1 + x^4}{x^4 \cdot \frac{x^2}{x^2} - x^4 \cdot \frac{x^2}{x^2}} = \lim_{x \to \pm\infty} \frac{1}{x^2 + 1} \to 1
\]

\[ f(x) = \frac{1 + x^4}{x^2 - x^4} = \frac{1 + x^4}{x^2(1-x^2)} = \frac{1 + x^4}{x^2(1+x)(1-x)} \]

denominator \(0\) when \(x = \pm 1, 0\).

\[
\lim_{x \to 1} \frac{1 + x^4}{x^2(1+x)(1-x)} \to 0^+ = \infty \] (depending on side

\[
\lim_{x \to 1^+} \frac{1 + x^4}{x^2(1+x)(1-x)} \to 0^+ = \infty
\]

\[
\lim_{x \to 0} \frac{1 + x^4}{x^2(1+x)(1-x)} \to 0^- = +\infty
\]

**horiz \(x\) at \(y = -1\)**

**vert at \(x = -1, 1, 0\)**