Math 221: Worksheet 2
Friday, September 1
Calculating Limits using the Limit Laws

Homework given in lecture yesterday: Section 2.3, #11, 13, 15, 17, 18, 20, 25, 26, 37, 39.

1. **Team Discussion:** Why might the technique of using tables to determine limits be misleading? Can you ever use a table of values to prove the limit of a function?

   **Answer:**
   Firstly, any table you make might not get close enough to actually determine the behavior of the function. For instance, it could look like it was approaching a certain value, but in the tiniest neighborhood go off to infinity.

   Even if you could make infinitely many table values however, it still might not give the actual behavior of the function. Consider \( \lim_{x \to 0} \sin \frac{\pi}{x} \). If you evaluated this function at 0.1, 0.01, 0.001, etc, the answer would be 0 for each of these values. However, we know that the values of \( \sin t \) oscillate as \( t \to \infty \), and the limit does not exist.

   You can never use a table of values to *prove* the limit of a function. The proof of this is very advanced, but it basically comes down to you not being able to list enough values to completely encompass the behavior of the real numbers (this idea is countable infinity vs. uncountable infinity). Fun stuff, but definitely beyond the scope of this course.

2. Determine whether each statement about limits is true or false. If it’s true, argue why. If it’s false, give an example that proves it wrong.

   (a) If \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist then \( \lim_{x \to a} \left[ f(x)g(x) \right] = \lim_{x \to a} f(x) \lim_{x \to a} g(x) \).

   **Answer:** False. Only if \( \lim_{x \to a} g(x) \neq 0 \).

   (b) If \( \lim_{x \to 6} \left[ f(x)g(x) \right] \) exists, then the limit is \( f(6)g(6) \).

   **Answer:** False. Consider \( f(x) = x - 6 \) and \( g(x) = \frac{1}{x - 6} \). \( \lim_{x \to 6} \left[ f(x)g(x) \right] = \lim_{x \to 6} \left[ (x - 6) \frac{1}{x - 6} \right] = 1 \), but \( f(6) = 0 \) and \( g(6) \) does not exist.

   (c) If \( \lim_{x \to 5} f(x) = 2 \) and \( \lim_{x \to 5} g(x) = 0 \), then \( \lim_{x \to 5} \frac{f(x)}{g(x)} \) does not exist.

   **Answer:** True. The limit doesn’t exist since \( \frac{f(x)}{g(x)} \) must approach infinity or negative infinity as \( x \) approaches \( 5 \), because the denominator approaches 0 and the numerator doesn’t.

   *Remember, infinity isn’t a number!!*

   (d) If \( \lim_{x \to 0} f(x) \) exists but \( \lim_{x \to 0} g(x) \) does not exist, then \( \lim_{x \to 0} \left[ f(x) + g(x) \right] \) does not exist.

   **Answer:** True. Suppose \( \lim_{x \to 0} \left[ f(x) + g(x) \right] \) in fact did exist. But then

   \[
   \lim_{x \to 0} g(x) = \lim_{x \to 0} \left[ \left( f(x) + g(x) \right) - f(x) \right] = \lim_{x \to 0} \left( f(x) + g(x) \right) - \lim_{x \to 0} f(x)
   \]

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by Limit Law 2, and \( \lim_{x \to 0} [f(x) + g(x)] - \lim f(x) \) exists, but that would mean \( \lim g(x) \) exists, which we already said it didn’t. So this is a contradiction, and \( \lim_{x \to 0} [f(x) + g(x)] \) does not exist.

If you didn’t get this whole proof, don’t worry. This is pretty advanced, but it’s a nice stretch concept, known as proof by contradiction.

(e) If \( \lim_{x \to 0} f(x) = \infty \) and \( \lim_{x \to 0} g(x) = \infty \), then \( \lim_{x \to 0} [f(x) - g(x)] = 0 \).

Answer: False. Consider \( f(x) = \frac{1}{x^2} \), and \( g(x) = \frac{1}{x^4} \). Both \( \lim_{x \to 0} f(x) = \infty \) and \( \lim_{x \to 0} g(x) = \infty \), but

\[
\lim_{x \to 0} [f(x) - g(x)] = \lim_{x \to 0} \left[ \frac{1}{x^2} - \frac{1}{x^4} \right] = \lim_{x \to 0} \left[ \frac{x^2}{x^4} - \frac{1}{x^4} \right] = \lim_{x \to 0} \frac{x^2 - 1}{x^4} = -\infty
\]

Remember that infinity isn’t a number, so these limits don’t exist, and therefore you can’t apply the limit laws.

(f) If \( \lim_{x \to a} f(x) = \lim_{x \to a} g(x) \) for all real numbers \( a \), and \( f(x) \) and \( g(x) \) are defined everywhere, then \( f = g \).

Answer: False. Consider \( g(x) \) that is the same as \( f(x) \) except at a single point of discontinuity. This single point of discontinuity won’t affect any of the limits of each function, so their limits will be equal at every point, but the functions themselves are not equal.

3. Carefully find the limit by applying a limit law at each step.

(a) \( \lim_{x \to 1} \left( \frac{x^7 - 2x^3 - 3x + 19}{x^4 - 4} \right)^2 \)

Answer: 25

\[
\lim_{x \to 1} \left( \frac{x^7 - 2x^3 - 3x + 19}{x^4 - 4} \right)^2 = \left( \lim_{x \to 1} \frac{x^7 - 2x^3 - 3x + 19}{x^4 - 4} \right)^2
\]

\[
= \left( \frac{\lim_{x \to 1} (x^7 - 2x^3 - 3x + 19)}{\lim_{x \to 1} (x^4 - 4)} \right)^2
\]

\[
= \left( \frac{\lim_{x \to 1} x^7 - \lim_{x \to 1} 2x^3 - \lim_{x \to 1} 3x + \lim_{x \to 1} 19}{\lim_{x \to 1} x^4 - \lim_{x \to 1} 4} \right)^2
\]

\[
= \left( \frac{1 - 2 - 3 + 19}{1 - 4} \right)^2
\]

\[
= \left( \frac{15}{-3} \right)^2 = (-5)^2 = 25
\]
(b) \( \lim_{x \to 2} \sqrt{x + 7x \cdot (x^3 - 2x)} \)

**Answer:** \( 16 \)

\[
\lim_{x \to 2} \sqrt{x + 7x \cdot (x^3 - 2x)} = \lim_{x \to 2} \sqrt{x + 7x} \cdot \lim_{x \to 2} (x^3 - 2x)
\]
\[
= \sqrt{\lim_{x \to 2} (x + 7x)} \cdot (\lim_{x \to 2} x^3 - \lim_{x \to 2} 2x)
\]
\[
= \sqrt{\lim_{x \to 2} x + \lim_{x \to 2} 7x} \cdot (\lim_{x \to 2} x^3 - \lim_{x \to 2} 2x)
\]
\[
= \sqrt{2 + 14} \cdot (8 - 4) = \sqrt{16} \cdot 4 = 4 \cdot 4 = 16
\]

4. Find the limits if they exist (without using graphing or tables). If they don’t exist explain why they don’t.

(a) \( \lim_{x \to 4^+} \frac{4 - x}{|4 - x|} \)

**Answer:** \( \lim_{x \to 4^+} \frac{4 - x}{|4 - x|} = \lim_{x \to 4^+} \frac{4 - x}{-(4 - x)} = \lim_{x \to 4^+} \frac{1}{1} = 1 \)

(b) \( \lim_{x \to 0} \frac{\sin(4x^2 + 3\pi/4)}{e^x - 1} \)

**Answer:** The limit does not exist, because the limit approaches \(-\infty\) from the left side and \(+\infty\) from the right. This is because at numbers close to 0, the numerator is close to \(\frac{1}{\sqrt{2}}\), which is positive, and the denominator approaches 0 through negative values as \(x\) approaches 0 from the right, and through positive values as \(x\) approaches 0 from the left.

(c) \( \lim_{x \to 3} \frac{7 - x}{3 - \sqrt{x} + 2} \)

**Answer:** 6

\[
\lim_{x \to 3} \frac{7 - x}{3 - \sqrt{x} + 2} = \lim_{x \to 3} \frac{7 - x}{3 - \sqrt{x} + 2} \cdot \frac{3 + \sqrt{x} + 2}{3 + \sqrt{x} + 2}
\]
\[
= \lim_{x \to 3} \frac{(7 - x)(3 + \sqrt{x} + 2)}{(3 - \sqrt{x} + 4)(3 + \sqrt{x} + 4)}
\]
\[
= \lim_{x \to 3} \frac{(7 - x)(3 + \sqrt{x} + 2)}{7 - x}
\]
\[
= \lim_{x \to 3} 3 + \sqrt{x} + 2 = 6
\]

(d) \( \lim_{x \to 11} \frac{1}{x - 11} \)

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Answer: \( \frac{1}{121} \)

\[
\lim_{x \to 11} \frac{x - \frac{1}{11}}{x - 11} = \lim_{x \to 11} \frac{x - \frac{1}{11}}{x - 11} \cdot \frac{11x}{11x} = \lim_{x \to 11} \frac{11 - x}{11x(x - 11)} = \lim_{x \to 11} \frac{-1}{11x} = -\frac{1}{121}
\]

(e) \( \lim_{h \to 0} \frac{(h - 7)^2 - 49}{h} \)

Answer: 14

\[
\lim_{h \to 0} \frac{(h + 7)^2 - 49}{h} = \lim_{h \to 0} \frac{h^2 + 14h + 49 - 49}{h} = \lim_{h \to 0} \frac{h^2 + 14h}{h} = \lim_{h \to 0} (h + 14) = 0 + 14 = 14
\]

(f) \( \lim_{x \to 2} \frac{3x - 6}{|x - 2|} \)

Answer: The limit does not exist.

\[
|x - 2| = \begin{cases} 
  x - 2 & \text{if } x - 2 \geq 0 \\
  -(x - 2) & \text{if } x - 2 \leq 0 
\end{cases} = \begin{cases} 
  x - 2 & \text{if } x \geq 2 \\
  -(x - 2) & \text{if } x \leq 2
\end{cases}
\]

So it’s easier to look at the one-sided limits:

\[
\lim_{x \to 2^+} \frac{3x - 6}{x - 2} = \lim_{x \to 2^+} \frac{3(x - 2)}{x - 2} = 3
\]

\[
\lim_{x \to 2^-} \frac{3x - 6}{-(x - 2)} = \lim_{x \to 2^-} \frac{3(x - 2)}{-(x - 2)} = -3
\]

So the left and right limits are different, and the limit does not exist.

(g) \( \lim_{x \to 3^+} \frac{\ln(x)}{\ln(10 - x^2)} \)

Answer: The limit tends to \(-\infty\). At numbers close to 3 (from the positive side), the numerator is close to \(\ln(3)\), which is positive, and the denominator approaches 0 through negative values as \(x\) approaches 3 from the right.

(h) \( \lim_{x \to -1} \sin\left(\frac{1}{x + 1}\right) \)

Answer: The limit does not exist. As \(x \to -1\), \(x + 1\) approaches 0, so \(\frac{1}{x+1}\) approaches infinity or negative infinity. As \(t \to \infty\), \(\sin t\) does not approach any one value - the sine function continues to oscillate between \(-1\) and \(1\). Thus, near \(x = -1\), \(\sin\left(\frac{1}{x+1}\right)\) oscillates between \(-1\) and \(1\).
Review: Trigonometry and the Unit Circle

1. Using the unit circle, deduce the identities:

(a) \( \cos(\pi - \theta) \) \textbf{Answer:} \( -\cos \theta \)
(b) \( \sin(\pi - \theta) \) \textbf{Answer:} \( \sin \theta \)
(c) \( \tan(\pi - \theta) \) \textbf{Answer:} \( \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{-\sin \theta}{-\cos \theta} = -\tan \theta \)
(d) \( \cos(\pi + \theta) \) \textbf{Answer:} \( -\cos \theta \)
(e) \( \sin(\pi + \theta) \) \textbf{Answer:} \( -\sin \theta \)
(f) \( \cos(\pi/2 - \theta) \) \textbf{Answer:} \( \sin \theta \)
(g) \( \sin(\pi/2 - \theta) \) \textbf{Answer:} \( \cos \theta \)
(h) \( \cos(\pi/2 + \theta) \) \textbf{Answer:} \( -\sin \theta \)
(i) \( \sin(\pi/2 + \theta) \) \textbf{Answer:} \( \cos \theta \)

2. Let \((x, y)\) be a point on the unit circle. If \( x = -\frac{4}{5} \), what are the possible values for \( y \)?

\textbf{Answer:} On the unit circle, \( x^2 + y^2 = 1 \), so if \( x = -4/5 \), then \( y = \pm 3/5 \).

3. Derive the identity \( \sin^2 \theta + \cos^2 \theta = 1 \).

\textbf{Answer:} Consider a right triangle with hypotenuse length \( c \). Let \( a \) be the length of the side adjacent to \( \theta \) and \( b \) be the length of the side opposite \( \theta \). Thus

\[ a^2 + b^2 = c^2 \Rightarrow \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \Rightarrow \left( \frac{a}{c} \right)^2 + \left( \frac{b}{c} \right)^2 = 1 \Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \]

Note that you can read the lengths of the sides directly as \( \sin \theta \) and \( \cos \theta \) if your hypotenuse is length 1 and you view this triangle on the unit circle.

4. Using the previous identity, derive the identities

(a) \( \tan^2 \theta + 1 = \sec^2 \theta \)

\textbf{Answer:} \( \sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \Rightarrow \tan^2 \theta + 1 = \sec^2 \theta \)

(b) \( 1 + \cot^2 \theta = \csc^2 \theta \)

\textbf{Answer:} \( \sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} \Rightarrow 1 + \cot^2 \theta = \csc^2 \theta \)

5. Which of these functions are even functions, and which are odd functions?

(a) \( \sin \theta \) \textbf{Answer:} odd
(b) \( \cos \theta \) \textbf{Answer:} even
(c) \( \tan \theta \) \textbf{Answer:} odd
(d) \( \csc \theta \) Answer: odd
(e) \( \sec \theta \) Answer: even 
(f) \( \cot \theta \) Answer: odd