1. At time \( t \) seconds, the velocity of an object is \( v(t) = t^2 + t + 1 \) m/s. Use the steps below to find the distance traveled by this object from \( t = 1 \) to \( t = 3 \).

(a) The distance traveled by an object is simply the area under its velocity function. Using \( v(t) \) graphed below, estimate the distance traveled.

![Graph of \( v(t) = t^2 + t + 1 \).](image)

Figure 1: Graph of \( v(t) = t^2 + t + 1 \).

(b) To find the area precisely, let's use a right Riemann sum. On the graph, draw rectangles to show how this would approximate the area. Does this overestimate or underestimate?
Recall \( \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^n f(x_k) \Delta x. \)

(c) If we use \( n \) intervals, what will the width \( \Delta t \) of each interval be? What will \( t_k \) (the right endpoint of the \( k \)th interval) be?

(d) Using the equation above, write the integral as the limit of a sum, and simplify \( v(t_k)\Delta t \) to a polynomial in \( k \), with coefficients involving \( n \).

(e) Distribute the summation over the terms in the polynomial, and pull the coefficients out of each summation.

(f) Using \( \sum_{i=1}^n i = \frac{n(n+1)}{2} \) and \( \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \), eliminate the summation terms.

(g) Now you should have a limit involving only \( n \). Solve this limit.

(h) Checking: Is your answer close to your estimate from part a)?

\[
\int_1^3 t^2 + t + 1 \, dt = \left[ \frac{t^3}{3} + \frac{t^2}{2} + t \right]_1^3 = \frac{9}{3} + \frac{9}{2} + 3 - \left( \frac{\frac{1}{3}}{3} + \frac{1}{2} + 1 \right) = 13 + \frac{8}{2} - \frac{1}{3} = 14 \frac{2}{3}
\]

(i) Discussion: The right Riemann sums overestimate the area under the curve, so why does the limit give us the precise integral? Why doesn’t it overestimate?
2. Let $f$ be the function shown below, and let $g(x) = \int_0^x f(t) \, dt$.

(a) Find $g(0), g(1), g(2), g(3), g(4), g(5),$ and $g(6)$.

\[ g(0) = 0, \quad g(1) = \frac{1}{2}, \quad g(2) = 0, \quad g(3) = -\frac{1}{2}, \quad g(4) = 0, \quad g(5) = \frac{3}{2}, \quad g(6) = 4 \]

(b) Where does $g$ have an absolute minimum on $[0,6]$ and where does $g$ have an absolute maximum on $[0,6]$?

\[ \min x \rightarrow g(3) = -\frac{1}{2}, \quad \max x \rightarrow g(6) = 4 \]

(c) Using the first part of the Fundamental Theorem of Calculus, what is $g'(x)$?

\[ g'(x) = \frac{d}{dx} \int_0^x f(t) \, dt = f(x). \]

(d) Where does $g$ have local maxima and minima, if any?

\[ \text{when } g'(x) = 0 \Rightarrow x = 1, \ x = 3 \quad \text{,} \quad 1 \ \text{is max,} \quad 3 \ \text{is min} \]

\[ f: \text{pos } \rightarrow \text{neg} \quad f: \text{neg } \rightarrow \text{pos} \]

(e) Where is $g$ concave up and where is $g$ concave down?

\[ g'' = f' > 0 \ \text{on} \ (0,1) \cup (3,6) \]

\[ < 0 \ \text{on} \ (1,3) \]

(f) On the graph above, sketch $g(x)$.
3. Comparing using the first and second parts of the Fundamental Theorem of Calculus to find Derivatives

Suppose that \( F(x) \) is an antiderivative of \( f(x) \), and \( a \) is a constant.

(a) Let \( G(x) = \int_{a}^{x^2} f(t) \, dt \). Using the second part of the Fundamental Theorem of Calculus, what is \( G'(x) \)? (It will include \( F(x) \)).

\[
G'(x) = F'(x^2) - F'(a)
\]

(b) What is \( F'(x) \)? What is \( F'(x^2) \)? What is \( F'(a) \)? (remember that \( a \) is a constant).

\[
F'(x) = f(x)
\]

\[
F'(x^2) = f(x^2) (2x)
\]

\[
F'(a) = 0
\]

(c) Use parts a) and b) to calculate \( G'(x) = \frac{d}{dx} \int_{a}^{x^2} f(t) \, dt \).

\[
G'(x) = F'(x^2) - F'(a)
\]

\[
= f(x^2) (2x) - 0 = 2x f(x^2)
\]

(d) Now find \( \frac{d}{dx} \int_{a}^{x^2} f(t) \, dt \) using only the first part of the Fundamental Theorem of Calculus and the Chain Rule.

\[
\frac{d}{dx} \int_{a}^{x^2} f(t) \, dt = \frac{d}{du} \left[ \int_{a}^{u} f(t) \, dt \right] \cdot \frac{du}{dx}
\]

\[
= f(u) \cdot 2x = f(x^2) \cdot 2x
\]

(e) **Discussion:** Check: are your answers the same? What are the similarities between using the first and the second part of the Fundamental Theorem of Calculus, and what are the differences? Which one do you prefer using?
4. Compute \( \frac{d}{dx} \int_{\cos x}^{x^2+2} \frac{1}{t^2+1} \, dt \) in two ways:

(a) Use the first part of the Fundamental Theorem of Calculus (break into two integrals, use integral rules to get them into the right form, and then use the FTC and the chain rule).

\[
= \frac{d}{dx} \left[ \int_0^{x^2+2} \frac{1}{t^2+1} \, dt + \int_0^{\cos x} \frac{1}{t^2+1} \, dt \right]
\]

\[
= \frac{d}{dx} \int_0^{x^2+2} \frac{1}{t^2+1} \, dt - \frac{d}{dx} \int_0^{\cos x} \frac{1}{t^2+1} \, dt
\]

\[
= \frac{du}{du} \int_0^{x^2+2} \frac{1}{t^2+1} \, dt + \frac{dv}{dv} \int_0^{\cos x} \frac{1}{t^2+1} \, dt \frac{du}{dx} - \frac{dv}{dx} \int_0^{\cos x} \frac{1}{t^2+1} \, dt
\]

\[
= \left( \frac{1}{u^2+1} \right) (2x) - \left( \frac{1}{v^2+1} \right) (-\sin x)
\]

\[
= \frac{1}{(x^2+2)^2+1} (2x) - \frac{1}{\cos^2 x + 1} (-\sin x)
\]

(b) Use the second part of the Fundamental Theorem of Calculus (like in parts a)-c) in problem 2).

\( f(t) = \frac{1}{t^2+1} \quad F(t) = \arctan t \)

\( G(x) = \int_{\cos x}^{x^2+2} \frac{1}{t^2+1} \, dt \)

\( G(x) = F(x^2+2) - F(\cos x) \)

\( G'(x) = f(x^2+2) (2x) - f(\cos x) (-\sin x) \)

\[
= \frac{1}{(x^2+2)^2+1} (2x) - \frac{1}{\cos^2 x + 1} (-\sin x)
\]

(c) **Discussion:** Which one do you prefer using now? Can you see how we get the second part of the FTC from the first part?
5. Evaluate the limits by first recognizing the sum as a Riemann sum for a function defined on \([0, 1]\).

(a) \[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^3}{n^4} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{i}{n} \right)^3 \frac{1}{n} \text{ n intervals, each length } \frac{1}{n}. \]

\[ = \int_{0}^{1} x^3 \, dx \]
\[ = \frac{1}{4} x^4 \bigg|_{0}^{1} = \frac{1}{4} - 0 = \frac{1}{4} \]

(b) \[ \lim_{n \to \infty} \frac{1}{n} \left( \sqrt[1]{\frac{1}{n}} + \sqrt[2]{\frac{2}{n}} + \cdots + \sqrt[n]{\frac{n}{n}} \right) \text{ (writing this in summation form may help).} \]

\[ \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt[1]{\frac{i}{n}} \left( \frac{1}{n} \right) \quad f(x) = \sqrt{x} \]
\[ = \int_{0}^{1} \sqrt{x} \, dx = \int_{0}^{1} x^{1/2} \, dx \]
\[ = \frac{2}{3} x^{3/2} \bigg|_{0}^{1} = \frac{2}{3} - 0 = \frac{2}{3} \]