Math 221: Worksheet 17
Areas, Distances, and the Definite Integral

Homework given in lecture yesterday: Read sections 5.1 (Areas and Distances) and 5.2 (The Definite Integral) very carefully. In section 5.1 do #3, 4, 13, 14, 15, 18, 22, 32. In section 5.2 do #3, 11, 21, 22, 29, 33, 36, 37, 41, 48, 49, 52, 53, 55, 57, 60.

1. We will show \( \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \).

   (a) Expand \( \sum_{i=1}^{n} i \) as a sum going from 1 to \( n \).

   (b) Expand \( \sum_{i=1}^{n} i \) as a sum going from \( n \) to 1.

   (c) Add the equations from parts a) and b), and simplify your result to find the value of \( \sum_{i=1}^{n} i \).

2. At time \( t \) seconds, the velocity of an object is \( v(t) = t^2 + t + 1 \) m/s. Use the steps below to find the distance traveled by this object from \( t = 1 \) to \( t = 3 \).

   (a) The distance traveled by an object is simply the area under its velocity function. Using \( v(t) \) graphed on page 3, estimate the distance traveled.

   (b) To find the area precisely, let’s use a right Riemann sum. On the graph, draw rectangles to show how this would approximate the area. Does this overestimate or underestimate?
Recall \[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^n f(x_k) \Delta x. \]

(c) If we use \( n \) intervals, what will the width \( \Delta t \) of each interval be? What will \( t_k \) (the right endpoint of the \( k \)th interval) be?

(d) Using the equation above, write the integral as the limit of a sum, and simplify \( v(t_k) \Delta t \) to a polynomial in \( k \), with coefficients involving \( n \).

(e) Distribute the summation over the terms in the polynomial, and pull the coefficients out of each summation.

(f) Using the result from question 1, and \( \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \), eliminate the summation terms.

(g) Now you should have a limit involving only \( n \). Solve this limit.
(h) Checking: Is your answer close to your estimate from part a)?

(i) Discussion: The right Riemann sums overestimate the area under the curve, so why does the limit give us the precise integral? Why doesn’t it overestimate?